2 Fast Physics

Imagine a summer's day. You are sunbathing by the side of a busy motorway while you wait for a pickup truck to rescue your car, which has broken down. All of a sudden, an irresponsible person throws a used drinks can out of their car window, and it heads in your direction. To make things worse, they were speeding at the time. Ouch.

The faster the car was going, the more it will hurt when the can hits you. This is because the can automatically takes up the speed the car was travelling at. Suppose the irresponsible person could throw the can at 10mph, and their car is going at 80mph. The speed of the can, as you see it, is 90mph if it was thrown forwards, and 70mph if it was thrown backwards.

To sum this up,

Velocity as measured by you = Velocity of car + Velocity of throwing

where we use velocities rather than speeds so that the directionality can be taken into account.

So far, this probably seems very obvious. However, let's extend the logic a bit further. Rather than a car, let us have a star, and in place of the drinks can, a beam of light. Many stars travel towards us at high speeds, and emit light as they do so. We can measure the speed of this light in a laboratory on Earth, and compare it with the speed of 'ordinary' light made in a stationary light bulb. And the worrying thing is that the two speeds are the same.

No matter how hard we try to change it, light always goes at the same speed.[4] This tells us that although our ideas of adding velocities are nice and straightforward, they are also wrong. In short, there is a problem with the Newtonian picture of motion. This problem is most obvious in the case of light, but it also occurs when anything else starts travelling very quickly.

While this is not the way Einstein approached the problem, it is our way into one of his early theories – the Special Theory of Relativity – and it is part of the Olympiad syllabus.

Before we go further and talk about what does happen when things go fast, please be aware of one thing. These observations will seem very

[4] Light does travel different speeds in different materials. However if the measurement is made in the same material (say, air or vacuum) the speed registered will always be the same, no matter what we do with the source.

weird if you haven't read them before. But don't dismiss relativity as nonsense just because it seems weird – it is a better description of Nature than classical mechanics – and as such it demands our respect and attention.

2.1 *The Principle of Relativity*

The theory of special relativity, like all theories, is founded on a premise or axiom. This axiom cannot be 'derived' – it is a guessed statement, which is the starting point for the maths and the philosophy. In the case of special relativity, the axiom must be helpful because its logical consequences agree well with experiments.

This principle, or axiom, can be stated in several ways, but they are effectively the same.

1. There is no method for measuring absolute (non-relative) velocity. The absolute speed of a car cannot be measured by any method at all. On the other hand, the speed of the car relative to a speed gun, the Earth, or the Sun can all be determined.

2. Since it can't be measured – there is no such thing as absolute velocity.

3. The 'laws of physics' hold in all non-accelerating laboratories[5], however 'fast' they may be going. This follows from statement 2, since if experiments only worked for one particular laboratory speed, that would somehow be a special speed, and absolute velocities could be determined relative to it.

4. Maxwell's theory of electromagnetism, which predicts the speed of light, counts as a law of physics. Therefore all laboratories will agree on the speed of light. It doesn't matter where or how the light was made, nor how fast the laboratory is moving.

2.2 *High Speed Observations*

In this section we are going to state what relativity predicts, as far as it affects simple observations. Please note that we are not deriving these statements from the principles in the last section, although this can be done. For the moment just try and understand what the statements mean. That is a hard enough job. Once you can use them, we shall then worry about where they come from.

[5] We say non-accelerating for a good reason. If the laboratory were accelerating, you would feel the 'inertial force', and thus you would be able to measure this acceleration, and indeed adjust the laboratory's motion until it were zero. However there is no equivalent way of measuring absolute speed.

2.2.1 Speeding objects look shortened in the direction of motion.

A metre stick comes hurtling towards you at high speed. With a clever arrangement of cameras and timers, you are able to measure its length as it passes you. If the stick's length is perpendicular to the direction of travel, you still measure the length as 1 metre.

However, if the stick is parallel to its motion, it will seem shorter to you. If we call the stick's actual length (as the stick sees it) as L_0, and the apparent length (as you measure it) L_a, we find

$$L_a = L_0 \sqrt{1 - \frac{u^2}{c^2}} \,, \qquad (1)$$

where u is the speed of the metre stick relative to the observer. The object in the square root appears frequently in relativistic work, and to make our equations more concise, we write

$$\gamma \equiv \frac{1}{\sqrt{1-(u/c)^2}} \qquad (2)$$

so that equation (1) appears in shorter form as

$$L_a = \frac{L_0}{\gamma} \,. \qquad (3)$$

2.2.2 Speeding clocks tick slowly

A second observation is that if a clock whizzes past you, and you use another clever arrangement of timers and cameras to watch it, it will appear (to you) to be going slowly.

We may state this mathematically. Let T_0 be a time interval as measured by our (stationary) clock, and let T_a be the time interval as we see it measured by the whizzing clock.

$$T_a = \frac{T_0}{\gamma} \qquad (4)$$

2.2.3 Slowing and shrinking go together

Equations (3) and (4) are consistent – you can't have one without the other. To see why this is the case, let us suppose that Andrew and Betty both have excellent clocks and metre sticks, and they wish to measure their relative speed as they pass each other. They must agree on the relative speed. Andrew times how long it takes Betty to travel along his metre stick, and Betty does the same.

The question is: how does Andrew settle his mind about Betty's calculation? As far as he is concerned, she has a short metre stick, and a slow clock – how can she possibly get the answer right! Very easily – providing that her clock runs 'slow' by the same amount that her metre stick is 'short'[6].

An experimental example may help clarify this. Muons are charged particles that are not stable, and decay with a half-life of 2μs. Because they are charged, you can accelerate them to high speeds using a large electric field in a particle accelerator. You can then measure how far they travel down a tube before decaying. Given that 'the laws of physics are the same in all reference frames', this must mean that muon and experimenter agree on the position in the tube at which the muon passes away.

The muon gets much further down the tube than a classical calculation would predict, however the reason for this can be explained in two ways:

- According to the experimenter, the muon is travelling fast, so it has a slow clock, and therefore lives longer – so it can get further.

- According to the muon, it still has a woefully short life, but the tube (which is whizzing past) is shorter so it appears to get further along in the 2μs.

For the two calculations to agree, the 'clock slowing' must be at the same rate as the 'tube shrinkage'.

2.2.4 Speeding adds weight to the argument

The most useful observation of them all, as far as the Olympiad syllabus is concerned is this: if someone throws a 1kg bag of sugar at you at high speed, and you (somehow) manage to measure its mass as it passes, you will register more than 1kg.

If the actual mass of the object is M_0, and the apparent mass is M_a, we find that

$$M_a = \gamma M_0 \,. \qquad (5)$$

The actual mass is usually called the 'rest mass' – in other words the mass as measured by an observer who is at rest with respect to the object.

[6] Note that 'slow' and 'short' are placed in quotation marks. Betty's clock and metre stick are not defective – however to Andrew they appear to be.

2.2.4.1 The Universal Speed Limit

This formula has important consequences. First of all, this is the origin of the 'universal speed limit', which is a well-known consequence of special relativity. This states that you will never measure the speed of an object (relative to you) as being greater than the speed of light.

Let us pause for a moment to see why. Suppose the object concerned is an electron in a particle accelerator (electrons currently hold the speed record on Earth for the fastest humanly accelerated objects). It starts at rest with a mass of about 10^{-30} kg. We turn on a large, constant electric field, and the electron starts to move relative to the accelerator. However, as it gets close to the speed of light, it starts to appear more massive. Therefore since our electric field (hence accelerating force) is constant, the electron's acceleration decreases. In fact, the acceleration tends to zero as time passes, although it never reaches zero exactly after a finite time. We are never able to persuade the electron to break the 'light-barrier', since when $u \to c$, $\gamma \to \infty$, and the apparent mass becomes very large (so the object becomes impossible to accelerate any further).

Please note that this does not mean that faster-than-light speeds can never be obtained. If we accelerate one electron to 0.6c Eastwards, and another to 0.6c Westwards, the approach speed of the two electrons is clearly superlumic (1.2c) as we measure it with Earth-bound speedometers. However, even in this case we find that the velocity of one of the electrons *as measured by the other* is still less than the speed of light. This is a consequence of our first observation – namely that relative velocities do not add in a simple way when the objects are moving quickly.

In fact the approach speed, as the electrons see it, is 0.882*c*. If you want to perform these calculations, the formula turns out to be

$$u_{AC} = \frac{u_{AB} + u_{BC}}{1 + u_{AB} u_{BC} / c^2}\,, \qquad (6)$$

where u_{AB} means the velocity of B as measured by A. Equation (6) only applies when all three relative velocities are parallel (or antiparallel).

2.2.4.2 Newton's Law of motion

Our second consequence is that we need to take great care when using Newton's laws. We need to remember that the correct form of the second law is

$$F = \frac{d}{dt}\mathrm{momentum} \qquad (7)$$

Why is care needed? Look closely for the trap – if the object speeds up, its mass will increase. Therefore the time derivative of the mass needs to be included as well as the time derivative of the velocity. We shall postpone further discussion until we have had a better look at momentum.

2.3 Relativistic Quantities

Now that we have mentioned the business of relativistic mass increase, it is time to address the relativistic forms of other quantities.

2.3.1 Momentum

Momentum is conserved in relativistic collisions, providing we define it as the product of the apparent mass and the velocity.

$$\mathbf{p} = \gamma m_0 \mathbf{u} \tag{7}$$

Notice that when you use momentum conservation in collisions, you will have to watch the γ factors. Since these are functions of the speed u, they will change if the speed changes.

2.3.2 Force

The force on a particle is the time derivative of its momentum. Therefore

$$\mathbf{F} = \frac{d}{dt}\mathbf{p} = m_0\left(\gamma \frac{d}{dt}\mathbf{u} + \mathbf{u}\frac{d\gamma}{dt}\right). \tag{8}$$

In the case where the speed is not changing, γ will stay constant, and the equation reduces to the much more straightforward $\mathbf{F}=\gamma m_0\mathbf{a}$. One example is the motion of an electron in a magnetic field.

2.3.3 Kinetic Energy

Now that we have an expression for force, we can integrate it with respect to distance to obtain the work done in accelerating a particle. As shown in section 1.1.1, this will give the kinetic energy of the particle. We obtain the result[7]

---

[7] If you wish to derive this yourself, here are the stages you need. Firstly, differentiate γ with respect to u to convince yourself that

$$\frac{d\gamma}{du} = \frac{u}{c^2\left(1-u^2/c^2\right)^{3/2}} = \frac{\gamma^3 u}{c^2} \quad \Rightarrow \quad \frac{du}{d\gamma} = \frac{c^2}{\gamma^3 u}.$$

Using this result, the derivation can be completed (see over the page):

$$K = (\gamma - 1)m_0 c^2 . \tag{9}$$

This states that the gain in energy of a particle when accelerated is equal to the gain in mass × c^2. From this we postulate that any increase in energy is accompanied by a change in mass. The argument works backwards too. When stationary, the particle had mass m_0. Surely therefore, it had energy m_0c^2 when at rest.

We therefore write the total energy of a particle as

$$E = K + m_0 c^2 = \gamma m_0 c^2 . \tag{10}$$

2.3.4 A Relativistic Toolkit

We can derive a very useful relationship from (10), (7) and the definition of γ:

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m_0^2 c^2 \left(c^2 - v^2\right) \\ &= \gamma^2 m_0^2 c^4 \left(1 - \left(\frac{v}{c}\right)^2\right). \\ &= m_0^2 c^4 \end{aligned} \tag{11}$$

This is useful, since it relates E and p without involving the nasty γ factor. Another equation which has no gammas in it can be derived by dividing momentum by total energy:

$$\frac{p}{E} = \frac{\gamma m_o u}{\gamma m_0 c^2} = \frac{u}{c^2} , \tag{12}$$

which is useful if you know the momentum and total energy, and wish to know the speed.

2.3.5 Tackling problems

If you have to solve a 'collision' type problem, avoid using speeds at all costs. If you insist on having speeds in your equations, you will also have gammas, and therefore headaches. So use the momenta and energies of the individual particles in your equations instead. Put more bluntly, you should write lots of 'p's, and 'E's, but no 'u's. Use the

---

$$\begin{aligned} K &= \int F dx = \int F u dt = \int m_0 u^2 \frac{d\gamma}{dt} dt + \int \gamma m_0 u \frac{du}{dt} dt \\ &= m_0 \int u \left(u + \gamma \frac{du}{d\gamma} \right) d\gamma = m_0 \int u \left(u + \frac{c^2}{\gamma^2 u} \right) d\gamma = m_0 \int u \frac{c^2}{u} d\gamma = \left[\gamma m_0 c^2 \right] \end{aligned}$$

conservation laws to help you. In relativistic work, you can always use the conservation of *E* – even in non-elastic collisions. The interesting thing is that in an inelastic collision, you will find the rest masses greater after the collision.

To obtain the values you want, you need an equation which relates E and p, and this is provided by (11). Notice in particular that the quantity $\left(\sum E\right)^2 - \left(\sum p\right)^2 c^2$ when applied to a group of particles has two things to commend it.

- Firstly, it is only a function of total energy and momentum, and therefore will remain the same before and after the collision.

- Secondly, it is a function of the rest masses (see equation 11) and therefore will be the same in all reference frames.

Finally, if the question asks you for the final speeds, use (12) to calculate them from the momenta and energies.

2.4 The Lorentz Transforms

The facts outlined above (without the derivations) will give you all the information you need to tackle International Olympiad problems. However, you may be interested to find out how the observations of section 2.2 follow from the general assumptions of section 2.1. A full justification would require a whole book on relativity, however we can give a brief introduction to the method here.

We start by stating a general problem. Consider two frames of reference (or co-ordinate systems) – Andrew's perspective (t,x,y,z), and Betty's perspective (t',x',y',z'). We assume that Betty is shooting past Andrew in the +x direction at speed *v*. Suppose an 'event' happens, and Andrew measures its co-ordinates. How do we work out the co-ordinates Betty will measure?

The relationship between the two sets of co-ordinates is called the Lorentz transformation, and this can be derived as shown below:

2.4.1 Derivation of the Lorentz Transformation

We begin with the assumption that the co-ordinate transforms must be linear. The reason for this can be illustrated by considering length, although a similar argument works for time as well. Suppose that Andrew has two measuring sticks joined end to end, one of length L1 and one of length L2. He wants to work out how long Betty reckons they are. Suppose the transformation function is T. Therefore Betty measures the first rod as T(L1) and the second as T(L2). She therefore will see that the total length of the rods is T(L1) + T(L2). This must also be equal to T(L1+L2), since L1+L2 is the length of the whole rod according to Andrew. Since T(L1+L2) = T(L1) +T(L2), the transformation function is linear.

We can now get to work. Let us consider Betty's frame of reference to be moving in the +*x* direction at speed *v*, as measured by Andrew. Betty will therefore see Andrew moving in her –*x* direction at the same speed. To distinguish Betty's co-ordinates from Andrew's, we give hers dashes.

Given the linear nature of the transformation, we write

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

where A, B, C and D are functions of the relative velocity +*v* (i.e. Betty's velocity as measured by Andrew).

There must also be an inverse transformation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{d} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

where *d* is the determinant of the first matrix.

Now this second matrix is in itself a transformation for a relative velocity –*v*, and therefore should be of a very similar form to the first matrix. We find that the only way we can ensure that there is symmetry between the two is to make the determinant equal to one (*d*=1). We shall therefore assume this from here on.

Next we consider what happens if *x'*=0. In other words we are tracing out Betty's motion as Andrew sees it. Therefore we must have *x*=*vt*. Using the first matrix, this tells us that *B*=-*vA*. A similar argument on the second matrix – where we must have *x'*=-*vt'* where Betty now watches Andrew's motion [*x*=*0*], gives –*Dv* = *B* = -*vA*. Therefore *A*=*D*.

We now have *B* and *D* expressed in terms of *A*, so the next job is to work out what *C* is. This can be done since we know that the determinant *AD* – *BC* = 1. Therefore we find that

$$C = \frac{1 - A^2}{vA}.$$

Summarizing, our matrix is now expressed totally in terms of the unknown variable *A*. We may calculate it by remembering that both Andrew and Betty will agree on the speed of travel, *c*, of a ray of light. Andrew will express this as $x=ct$, Betty would say $x'=ct'$, but both must be valid ways of describing the motion. Therefore

$$\begin{pmatrix} ct' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} ct \\ t \end{pmatrix}$$

$$c = \frac{ct'}{t'} = \frac{Ac+B}{Cc+D}$$

$$(Cc+D)c = Ac+B$$

$$Cc^2 = B \text{ since A} = \text{D}$$

$$\left(\frac{1\text{ - A}^2}{vA}\right)c^2 = -vA$$

$$\left(1-A^2\right)c^2 = -v^2A^2$$

$$A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

This concludes our reasoning, and gives the Lorentz transforms (after a little algebra to evaluate *C*) as:

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

$$x = \gamma(x' + vt') \quad .$$

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

$$\gamma \equiv \frac{1}{\sqrt{\left(1-\left[\frac{v}{c}\right]^2\right)}}$$

We have not considered any other dimensions here, however the transformation here is easy since Andrew and Betty agree on all lengths in the *y* and *z* directions. In other words $y'=y$, $z'=z$. This is a necessary consequence of the principle of relativity: the distance between the ends of a rod held perpendicular to the direction of motion can be measured simultaneously in all frames of reference. If this agreed measurement was different to that of an identical rod in a different frame, the observers would be able to work out which of them was 'moving' and which of them was still.

2.4.2 Using the Lorentz Transforms

Having these transforms at our disposal, we can now derive the 'shrinking rod' and 'slowing clock' equations.

Suppose Betty is holding a stick (of length L) parallel to the x-axis. We want to know how long Andrew thinks it is. To measure it, he will measure where the ends of the rod are at a particular moment, and will then measure the distance between these points. Clearly the two positions need to be measured simultaneously in his frame of reference, and thus t is the same for both measurements. We know from that Betty thinks it has length L, and therefore $\Delta x'=L$. Using the first of the Lorentz equations (the one which links x', x and t), and remembering that t is the same for both measurements,

$$\Delta x' = \gamma \Delta x$$
$$L_{apparent} = \frac{L}{\gamma}.$$

Similarly we may show how a clock appears to slow down. Betty is carrying the clock, so it is stationary with respect to her, and x' (her measurement of the clock's position) will therefore be constant. The time interval shown on Betty's clock is $\Delta t'$, while Andrew's own clock will measure time Δt. Here $\Delta t'$ is the time Andrew sees elapsing on Betty's clock, and as such is equal to T_{apparent}. Using the fourth Lorentz equation (the one with x', t and t' in it), and remembering that x' remains constant, we have

$$\Delta t = \gamma \Delta t'$$
$$T_{apparent} = \frac{T}{\gamma}.$$

2.4.3 Four Vectors

The Lorentz transforms show you how to work out the relationships between the (t,x,y,z) co-ordinates measured in different frames of reference. We describe anything that transforms in the same way as a four vector, although strictly speaking we use (ct,x,y,z) so that all the components of the vector have the same units. Three other examples of four vectors are:

- $(\gamma c, \gamma u_x, \gamma u_y, \gamma u_z)$ is called the *four velocity* of an object, and is the derivative of (ct, x,y,z) with respect to the proper time τ. Proper time is the time elapsed as measured in the rest frame of the object $t=\gamma\tau$.

- (mc, p_x, p_y, p_z) the momentum four vector. Here *m* is equal to γm_0. This must be a four vector since it is equal to the rest mass multiplied by the four velocity (which we already know to be a four vector).

- $(\omega/c, k_x, k_y, k_z)$ the wave four vector, where ω is the angular frequency of the wave ($\omega=2\pi f$), and **k** is a vector which points in the direction the wave is going, and has magnitude $2\pi/\lambda$. This can be derived from the momentum four vector in the case of a photon, since the momentum and total energy of a photon are related by E=pc, and the quantum theory states that $E=hf=h\omega/2\pi$ and $p=h/\lambda=hk/2\pi$.

It also turns out that the dot product of any two four-vectors is 'frame-invariant' – in other words all observers will agree on its value. The dot product of two four-vectors is slightly different to the conventional dot product, as shown below:

$$(ct, x, y, z) \bullet (ct, x, y, z) \equiv x^2 + y^2 + z^2 - (ct)^2 .$$

Notice that we subtract the product of the first elements.

The dot product of the position four vector with the wave four vector gives

$$(ct, x, y, z) \bullet (\omega/c, k_x, k_y, k_z) \equiv \mathbf{k} \bullet \mathbf{r} - \omega t .$$

Now this is the phase of the wave, and since all observers must agree whether a particular point is a peak, a trough or somewhere in between, then the phase must be an invariant quantity. Accordingly, since (ct,x,y,z) makes this invariant when 'dotted' with $(\omega/c, k_x, k_y, k_z)$, it follows that $(\omega/c, k_x, k_y, k_z)$ must be a four vector too.

2.5 Questions

1. Work out the relativistic γ factor for speeds of 1%, 50%, 90% and 99% of the speed of light.

2. Work out the speeds needed to give γ factors of 1.0, 1.1, 2.0, 10.0.

3. A muon travels at 90% of the speed of light down a pipe in a particle accelerator at a steady speed. How far would you expect it to travel in 2μs (a) without taking relativity into account, and (b) taking relativity into account?

4. A particle with rest mass *m* and momentum *p* collides with a stationary particle of mass *M*. The result is a single new particle of rest mass *R*. Calculate *R* in terms of *p* and *M*.

5. The principal runway at the spaceport on Arcturus-3 has white squares of side length 10m painted on it. A set of light sensors on the base of a spacecraft can take a 'picture' of the whole runway at the same time. What will the squares look like in the image if the spacecraft is passing the runway at a very high speed? Each sensor takes a picture of the runway directly underneath it, so you do not need to take into account the different times taken by light to reach the sensors from different parts of the runway.

6. When an electron is accelerated through a voltage V, its kinetic energy is given by eV where e is the size of the charge on the electron and is equal to 1.6×10^{-19}C. Taking the mass of the electron to be 9.1×10^{-31}kg, work out (a) the kinetic energy and speed of the electron when V=511kV (b) the kinetic energy and speed when V=20kV (c) the percentage error in the kinetic energy for V=20kV when calculated using the non-relativistic equation $\frac{1}{2}mu^2$.

7. Prove that the kinetic energy of a particle of rest mass m and speed u is given by $\frac{1}{2}mu^2$ if the speed is small enough in comparison to the speed of light. Work out the speed at which the non-relativistic calculation would be in error by 1%.

8. Suppose a spacecraft accelerates with constant acceleration a (as measured by the spacecraft's onboard accelerometers). At t=0 it is at rest with respect to a planet. Work out its speed relative to the planet as a function of time (a) as measured by clocks on the spacecraft, and (b) as measured by clocks on the planet. Note that the instantaneous speed of the craft relative to the planet will be agreed upon by spacecraft and planet.