1 Linear Mechanics

1.1 *Motion in a Line*

1.1.1 The Fundamentals

1.1.1.1 Kinematics

Mechanics is all about motion. We start with the simplest kind of motion – the motion of small dots or particles. Such a particle is described completely by its mass (the amount of stuff it contains) and its position. There is no internal structure to worry about, and as for rotation, even if it tried it, no-one would see. The most convenient way of labelling the position is with a vector **r** showing its position with respect to some convenient agreed stationary point.

If the particle is moving, its position will change. If its speed and direction are steady, then we can write its position after time t as

$$\mathbf{r} = \mathbf{s} + \mathbf{u}t,$$

where **s** is the starting point (the position of the particle at t=0) and **u** as the change in position each second – otherwise known as the velocity. If the velocity is not constant, then we can't measure it by seeing how far the object goes in one second, since the velocity will have changed by then. Rather, we say that u how far the object would go in one second if the speed or direction remained unchanged that long. In practice, if the motion remains constant for some small time (called δt), and during this small time, the particle's position changes $\delta\mathbf{r}$, then the change in position if this were maintained for a whole second (otherwise known as the velocity) is

$$\mathbf{u} = \delta\mathbf{r} \times \text{number of } \delta t \text{ periods in one second} = \delta\mathbf{r} \div \delta\text{t}.$$

Similarly, if the velocity is changing, we define the acceleration as the change in velocity each second (if the rate of change of acceleration were constant. Accordingly, our equation for acceleration becomes

$$\mathbf{a} = \delta\mathbf{u} \div \delta t.$$

Hopefully, it is apparent that as the motion becomes more complex, and the δt periods need to be made shorter and shorter, we end up with the differential equations linking position, velocity and acceleration:

$$\begin{array}{ll} \mathbf{u} = \dfrac{d}{dt}\mathbf{r} & \mathbf{r} = \int \mathbf{u}\, dt \\ \mathbf{a} = \dfrac{d}{dt}\mathbf{u} & \mathbf{u} = \int \mathbf{a}\, dt \end{array}.$$

1.1.1.2 Dynamics

Now we have a way of describing motion, we need a way of predicting or explaining the motion which occurs – changing our question from ‘what is happening?’ to ‘why?’ and our explanation is going to involve the activity of forces. What do forces do to an object?

The first essential point is that forces are only needed to change (not maintain) motion. In other words – unless there is a change of velocity, no force is needed. But how much force is needed?

Newton made the assumption (which we find to be helpful and true) that the force causes a change in what he called the ‘motion’ –we now call it momentum. Suppose an object has mass m and velocity $\mathbf{u}$ (we shall clarify what we mean by mass later) – then its momentum is equal to $m\mathbf{u}$, and is frequently referred to by physicists by the letter $\mathbf{p}$. Newton’s second law states that if a constant force $\mathbf{F}$ is applied to an object for a short time δt, then the change in the momentum is given by $\mathbf{F}\,\delta t$. In differential notation $d(m\mathbf{u})/dt = \mathbf{F}$.

In the case of a single object of constant mass it follows that

$$\mathbf{F} = \frac{d(m\mathbf{u})}{dt} = m\frac{d\mathbf{u}}{dt} = m\mathbf{a}\ .$$

His next assumption tells us more about forces and allows us to define ‘mass’ properly. Imagine two bricks are being pulled together by a strong spring. The brick on the left is being pulled to the right, the brick on the right is being pulled to the left.

Newton assumed that the force pulling the left brick rightwards is equal and opposite to the force pulling the right brick leftwards. To use more mathematical notation, if the force on block no.1 caused by block no.2 is called $\mathbf{f}_{12}$, then $\mathbf{f}_{12}=-\mathbf{f}_{21}$. If this were not the case, then if we looked at the bricks together as a whole object, the two internal forces would not cancel out, and there would be some ‘left over’ force which could accelerate the whole object.[1]

It makes sense that if the bricks are identical then they will accelerate together at the same rate. But what if they are not? This is where Newton’s second law is helpful. If the resultant force on an object of

[1] If you want to prove that this is ridiculous, try lifting a large bucket while standing in it.

constant mass equals its mass times its acceleration, and if the two forces are equal and opposite, we say

$$\begin{aligned} \mathbf{f}_{12} &= -\mathbf{f}_{21} \\ m_1\mathbf{a}_1 &= -m_2\mathbf{a}_2\ , \\ \frac{a_1}{a_2} &= \frac{m_2}{m_1} \end{aligned}$$

and so the 'more massive' block accelerates less. This is the definition of mass. Using this equation, the mass of any object can be measured with respect to a standard kilogram. If a mystery mass experiences an acceleration of 2m/s^2 while pushing a standard kilogram in the absence of other forces, and at the same time the kilogram experiences an acceleration of 4m/s^2 the other way, then the mystery mass must be 2kg.

When we have a group of objects, we have the option of applying Newton's law to the objects individually or together. If we take a large group of objects, we find that the total force

$$\mathbf{F}_{\text{total}} = \sum_i \mathbf{F}_i = \frac{d}{dt}\sum_i m_i \mathbf{u}_i$$

changes the total momentum (just like the individual forces change the individual momenta). Note the simplification, though – there are no $\mathbf{f}_{ij}$ in the equation. This is because $\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0$, so when we add up the forces, the internal forces sum to zero, and the total momentum is only affected by the external forces $\mathbf{F}_i$.

1.1.1.3 Energy and Power

Work is done (or energy is transferred) when a force moves something. The amount of work done (or amount of energy transformed) is given by the dot product of the force and the distance moved.

$$W = \mathbf{F} \bullet \mathbf{r} = F\, r \cos\theta \qquad (1)$$

where θ is the angle between the force vector **F** and the distance vector **r**. This means that if the force is perpendicular to the distance, there is no work done, no energy is transferred, and no fuel supply is needed.

If the force is constant in time, then equation (1) is all very well and good, however if the force is changing, we need to break the motion up into little parts, so that the force is more or less constant for each part. We may then write, more generally,

$$\delta W = \mathbf{F} \bullet \delta\mathbf{r} = F\, \delta r \cos\theta \qquad (1a)$$

Two useful differential equations can be formed from here.

1.1.1.4 Virtual Work

From equation (1a) it is clear that if the motion is in the direction of the force applied to the object (i.e. θ=0), then

$$\frac{\delta W}{\delta r} = F \ ,$$

where W is the work done on the object. Accordingly, we can calculate the force on an object if we know the energy change involved in moving it. Let's give an example.

An electron (with charge *q*) is forced through a resistor (of length *L*) by a battery of voltage *V*. As it goes through, it must lose energy *qV*, since *V* is the energy loss per coulomb of charge passing through the resistor. Therefore, assuming that the force on the electron is constant (which we assume by the symmetry of the situation), then the force must be given by $\delta W / \delta d = qV / L$. If we define the electric field strength to be the force per coulomb of charge (*F/q*), then it follows that the electric field strength $E = V/L$.

So far, we have ignored the sign of *F*. It can not have escaped your attention that things generally fall downwards – in the direction of decreasing [gravitational] energy. In equations (1) and (1a), we used the vector **F** to represent the externally applied force we use to drag the object along. In the case of lifting a hodful of bricks to the top of a wall, this force will be directed upwards. If we are interested in the force of gravity **G** acting on the object (whether we drag it or not), this will be in the opposite direction. Therefore **F** = –**G**, and

$$\delta W = -\,\mathbf{G} \bullet \delta\mathbf{r}, \qquad (1b)$$

$$G = -\frac{\delta W}{\delta r} \, .$$

In other words, if an object can lose potential energy by moving from one place to another, there will always be a force trying to push it in this direction.

1.1.1.5 Power

Another useful equation can be derived if we differentiate (1a) with respect to time. The rate of 'working' is the power P, and so

$$P = \frac{\delta W}{\delta t} = \frac{\mathbf{F} \bullet \delta r}{\delta t} = \mathbf{F} \bullet \frac{\delta r}{\delta t} \, .$$

As we let the time period tend to zero, $\delta\mathbf{r}/\delta t$ becomes the velocity, and so we have:

$$P = \mathbf{F} \bullet \mathbf{u} = F\,u \cos\theta \qquad (2)$$

where θ is now best thought of as the angle between force and direction of motion. Again we see that if the force is perpendicular to the direction of motion, no power is needed. This makes sense: think of a bike going round a corner at constant speed. A force is needed to turn the corner - that's why you lean into the bend, so that a component of your weight does the job. However no work is done – you don't need to pedal any harder, and your speed (and hence kinetic energy) does not change.

Equation (2) is also useful for working out the amount of fuel needed if a working force is to be maintained. Suppose a car engine is combating a friction force of 200N, and the car is travelling at a steady 30m/s. The engine power will be 200N × 30m/s = 6 kW.

Our equation can also be used to derive the kinetic energy. Think of starting the object from rest, and calculating the work needed to get it going at speed *U*. The force, causing the acceleration, will be **F**=m**a**. The work done is given by

$$\begin{aligned} W &= \int P\,dt = \int \mathbf{F} \bullet \mathbf{v}\,dt = \int m\frac{d\mathbf{v}}{dt} \bullet \mathbf{v}\,dt \\ &= \int m\mathbf{v} \bullet d\mathbf{v} = \left[\tfrac{1}{2}mv^2\right]_0^U = \tfrac{1}{2}mU^2 \end{aligned} \qquad (3)$$

although care needs to be taken justifying the integration stage in the multi-dimensional case.[2]

1.1.2 Changing Masses

The application of Newton's Laws to mechanics problems should pose you no trouble at all. However there are a couple of extra considerations which are worth thinking about, and which don't often get much attention at school.

The first situation we'll consider is when the mass of a moving object changes. In practice the mass of any self-propelling object will change as it uses up its fuel, and for accurate calculations we need to take this into account. There are two cases when this *must* be considered to get the answer even roughly right – jet aeroplanes and rockets. In the case of rockets, the fuel probably makes up 90% of the mass, so it must not be ignored.

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[2] The proof is interesting. It turns out that $\mathbf{v} \bullet \mathbf{dv} \equiv |\mathbf{v}||\mathbf{dv}|\cos\theta = v\,dv$ since the change in *speed* dv is equal to |**dv**| cosθ where **dv** (note the bold type) is the vector giving the change in velocity.

Changing mass makes the physics interesting, because you need to think more carefully about Newton's second law. There are two ways of stating it – either

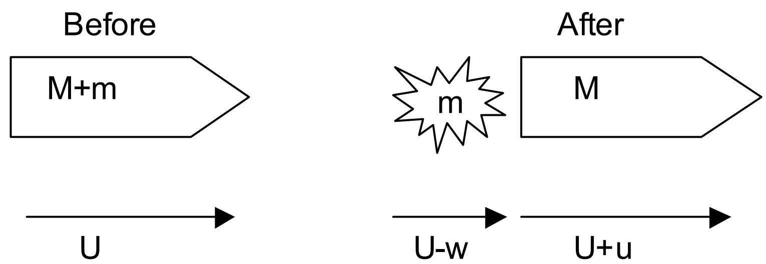
(i) Force on an object is equal to the rate of change of its momentum
(ii) Force on an object is equal to mass × acceleration

The first says $F = d(mu)/dt = m\dot{u} + \dot{m}u = ma + \dot{m}u$, whereas the second simply states $F=ma$. Clearly they can't both be correct, since they are different. Which is right? The first: which was actually the way Newton stated it in the first place! The good old $F=ma$ will still work – but you have to break the rocket into parts (say grams of fuel) – so that the rocket loses parts, but each part does not lose mass – and then apply $F=ma$ to each individual part. However if you want to apply a law of motion to the rocket as a whole, you have to use the more complicated form of equation.

This may be the first time that you encounter the fact that momentum is a more 'friendly' and fundamental quantity to work with mathematically than force. We shall see this in a more extreme form when looking at special relativity.

Let us now try and calculate how a rocket works. We'll ignore gravity and resistive forces to start with, and see how fast a rocket will go after it has burnt some fuel. To work this out we need to know two things – the exhaust speed of the combustion gas (*w*), which is always measured relative to the rocket; and the rate at which the motor burns fuel (in kg/s), which we shall call α.

We'll think about one part of the motion, when the rocket starts with mass (*M*+*m*), burns mass m of fuel, where m is very small, and in doing so increases its speed from *U* to *U*+*u*. This is shown below in the diagram.



Notice that the velocity of the burnt fuel is *U*-*w*, since *w* is the speed at which the combustion gas leaves the rocket (backwards), and we need to take the rocket speed *U* into account to find out how fast it is going relative to the ground.

Conservation of momentum tells us that

$$(M+m)\ U = m\ (U-w) + M\ (U+u)$$

so $$m\,w = M\,u. \qquad (4)$$

We can integrate this expression for *u* to evaluate the total change in speed after burning a large amount of fuel. We treat the *u* (change in *U*) as an infinitesimal calculus *dU*, and the *m* as a calculus *–dM*. Notice the minus sign – clearly the rocket must lose mass as fuel is burnt. Equation (4) now tells us

$$-w\frac{dM}{M} = dU \qquad (5)$$

This can be integrated to give

$$\begin{aligned} -w\int\frac{1}{M}dM &= \int dU \\ -w[\ln M] &= [U] \\ U_{final} - U_{initial} &= w\ln\left(\frac{M_{initial}}{M_{final}}\right) \end{aligned} \qquad (6)$$

This formula (6) is interesting because it tells us that in the absence of other forces, the gain in rocket speed depends *only* on the fraction of rocket mass that is fuel, and the exhaust speed.

In this calculation, we have ignored other forces. This is not a good idea if we want to work out the motion at blast off, since the Earth's gravity plays a major role! In order to take this, or other forces, into account, we need to calculate the thrust force of the rocket engine – a task we have avoided so far.

The thrust can be calculated by applying $F=ma$ to the (fixed mass) rocket M in our original calculation (4). The acceleration is given by $dU/t = u/t$, where t is the time taken to burn mass m of fuel. The thrust is

$$T = M\times\frac{u}{t} = M\times\frac{mw}{Mt} = \frac{mw}{t} = w\times\frac{m}{t} = w\alpha \qquad (7)$$

given by the product of the exhaust speed and the rate of burning fuel. For a rocket of total mass M to take off vertically, T must be greater than the rocket's weight Mg. Therefore for lift off to occur at all we must have

$$w\alpha > Mg\ . \qquad (8)$$

This explains why 'heavy' hydrocarbon fuels are nearly always used for the first stage of liquid fuel rockets. In the later stages, where absolute thrust is less important, hydrogen is used as it has a better 'kick per kilogram' because of its higher exhaust speed.

1.1.3 Fictitious Forces

Fictitious forces do not exist. So why do we need to give them a moment's thought? Well, sometimes they make our life easier. Let's have a couple of examples.

1.1.3.1 Centrifugal Force

You may have travelled in one of those fairground rides in which everyone stands against the inside of the curved wall of a cylinder, which then rotates about its axis. After a while, the floor drops out – and yet you don't fall, because you're "stuck to the side". How does this work?

There are two ways of thinking about this. The first is to look at the situation from the stationary perspective of a friend on the ground. She sees you rotating, and knows that a centripetal force is needed to keep you going round – a force pointing towards the centre of the cylinder. This force is provided by the wall, and pushes you inwards. You feel this strongly if you're the rider! And by Newton's third law it is equally true that you are pushing outwards on the wall, and this is why you feel like you are being 'thrown out'.

While this approach is correct, sometimes it makes the maths easier if you analyse the situation from the perspective of the rider. Then you don't need to worry about the rotation! However in order to get the working right you have to include an outwards force – to balance the inward push of the wall. If this were not done, the force from the walls would throw you into the central space. The outward force is called the centrifugal force, and is our first example of a fictitious force. It doesn't *really* exist, unless you are working in a rotating reference frame, and insist that you are at rest.

The difference between the two viewpoints is that in one case the inward push of the wall provides the centripetal acceleration. In the other it opposes the centrifugal force - giving zero resultant, and keeping the rider still. Therefore the formulae used to calculate centripetal force also give the correct magnitude for centrifugal force. The two differences are:

(i) Centrifugal force acts outwards, centripetal force acts inwards

(ii) Centrifugal force is *only* considered if you are assuming that the cylinder is at rest (in the cylinder's reference frame). On the other hand, you *only* have centripetal accelerations if you do treat the cylinder as a moving object and work in the reference frame of a stationary observer.

This example also shows that fictitious forces generally act in the opposite direction to the acceleration that is being 'ignored'. Here the acceleration is an inward centripetal acceleration, and the fictitious centrifugal force points outward.

1.1.3.2 Inertial Force

The second example we will look at is the motion of a lift (elevator) passenger. You know that you 'feel heavier' when the lift accelerates upwards, and 'feel lighter' when it accelerates downwards. Therefore if you want to simplify your maths by treating the lift car as a stationary box, you must include an extra downward force when the lift is actually accelerating upwards, and vice-versa. This fictitious force is called the inertial force. We see again that it acts in the opposite direction to the acceleration we are trying to ignore.

We shall look more closely at this situation, as it is much clearer mathematically.

Suppose we want to analyse the motion of a ball, say, thrown in the air in a lift car while it is accelerating upwards with acceleration $\mathbf{A}$. We use the vector $\mathbf{a}$ to represent the acceleration of the ball as a stationary observer would measure it, and $\boldsymbol{a}'$ to represent the acceleration as measured by someone in the lift. Therefore, $\boldsymbol{a} = \boldsymbol{A} + \boldsymbol{a}'$. Now this ball won't simply travel in a straight line, because forces act on it. Suppose the force on the ball is $\mathbf{F}$. We want to know what force $\mathbf{F}'$ is needed to get the right motion if we assume the lift to be at rest.

Newton's second law tells us that $\mathbf{F}=m\mathbf{a}$, if m is the mass of the ball. Therefore $\mathbf{F}=m(\mathbf{A}+\mathbf{a}')$, and so $\mathbf{F}-m\mathbf{A} = m\mathbf{a}'$. Now the force $\mathbf{F}'$ must be the force needed to give the ball acceleration $\mathbf{a}'$ (the motion relative to the lift car), and therefore $\mathbf{F}'=m\mathbf{a}'$. Combining these equations gives

$$\mathbf{F}' = \mathbf{F} - m\mathbf{A}. \tag{9}$$

In other words, if working in the reference frame of the lift, you need to include not only the forces which are really acting on the ball (like gravity), but also an extra force $-m\mathbf{A}$. This extra force is the inertial force.

Let us continue this line of thought a little further. Suppose the only force on the ball was gravity. Therefore $\mathbf{F}=m\mathbf{g}$. Notice that

$$\mathbf{F}' = \mathbf{F} - m\mathbf{A} = m\,(\mathbf{g}-\mathbf{A}) \tag{10}$$

and therefore if $\mathbf{g}=\mathbf{A}$ (that is, the lift is falling like a stone, because some nasty person has cut the cable), $\mathbf{F}'=\mathbf{0}$. In other words, the ball behaves as if no force (not even gravity) were acting on it, at least from the point of view of the unfortunate lift passengers. This is why weightlessness is experienced in free fall.

A similar argument can be used to explain the weightlessness of astronauts in orbiting spacecraft. As stationary observer (or a physics teacher) would say that there is only one force on the astronauts – gravity, and that this is just the right size to provide the centripetal force. The astronaut's perspective is a little different. He (or she) experiences two forces – gravity, and the fictitious centrifugal force. These two are equal and opposite, and as a result they add to zero, and so the astronaut feels just as weightless as the doomed lift passengers in the last paragraph.

1.2 *Going Orbital*

1.2.1 We have the potential

We shall now spend a bit of time reviewing gravity. This is a frequent topic of Olympiad questions, and is another area in which you should be able to do well with your A-level knowledge.

Gravitation causes all objects to attract all other objects. To simplify matters, we start with two small compact masses. The size of the force of attraction is best described by the equation

$$F_r = -\frac{GMm}{R^2} \tag{11}$$

Here G is the Gravitational constant (6.673×10^{-11} $\mathrm{Nm^2/kg^2}$), M is the mass of one object (at the origin of coordinates), and m is the mass of the other. The equation gives the force experienced by the mass m. Notice the 'r' subscript and the minus sign – the force is radial, and directed inwards toward the origin (where the mass M is).

It is possible to work out how much work is needed to get the mass m as far away from M as possible. We use integration

$$\int_R^\infty \mathbf{F}\bullet\mathbf{dx} = \int_R^\infty -F_r\,dr = \int_R^\infty \frac{GMm}{r^2}\,dr = -\left[\frac{GMm}{r}\right]_R^\infty = \frac{GMm}{R}.$$

Notice the use of $-F_r$ in the second stage. In order to separate the masses we use a force $\mathbf{F}$ which acts in opposition to the gravitational attraction F_r. The equation gives the amount of work done by this force as it pulls the masses apart.

We usually define the zero of potential energy to be when the masses have nothing to do with each other (because they are so far away). Accordingly, the potential energy of the masses m and M is given by

$$E(R) = -\frac{GMm}{R}. \tag{12}$$

That is, GMm/R joules below zero energy. Notice that

$$F_r(R) = -\frac{dE(R)}{dR}. \tag{13}$$

This is a consequence of the definition of work as $W = \int \mathbf{F} \bullet \mathbf{dx}$, and is generally true. It is useful because it tells us that a forces always point in the direction of decreasing energy.

The potential energy depends on the mass of both objects as well as the position. The gravitational potential *V(R)* is defined as the energy per unit mass of the second object, and is given by

$$V(R) \equiv \lim_{m \to 0} \frac{E(R,m)}{m} = -\frac{GM}{R}. \tag{14}$$

Accordingly, the potential is a function only of position. The zero limit on the mass m is needed (in theory) to prevent the small mass disturbing the field. In practice this will not happen if the masses are fixed in position. To see the consequences of breaking this rule, think about measuring the Earth's gravitational field close to the Moon. If we do this by measuring the force experienced by a 1kg mass, we will be fine. If we do it by measuring the force experienced by a 10^{28}kg planet put in place for the job, we will radically change the motion of Earth and Moon, and thus affect the measurement.

In a similar way, we evaluate the gravitational field strength as the force per kilogram of mass. Writing the field strength as g gives

$$g = -\frac{MG}{R^2} \tag{15}$$

and equation (13) may be rewritten in terms of field and potential as

$$g(R) = -\frac{dV}{dR}. \tag{16}$$

1.2.2 Orbital tricks

There is a useful shortcut when doing problems about orbits. Suppose that an object of mass m is orbiting the centre of co-ordinates, and experiences an attractive force $F_r = -Ar^n$, where A is some constant. Therefore $n=-2$ for gravity, and we would have $n=+1$ for motion of a particle attached to a spring (the other end fixed at the origin).

If the object is performing circular orbits, the centripetal acceleration will be u^2/R where R is the radius of the orbit. This is provided by the attractive force mentioned, and so:

$$\frac{mu^2}{R} = -F_r = AR^n$$
$$\frac{mu^2}{2} = \frac{AR^{n+1}}{2} \tag{17}$$

Now the potential energy E(R) is such that $dE/dR = -F_r = AR^n$, so

$$E(R) = \frac{AR^{n+1}}{n+1} \tag{18}$$

if we take the usual convention that E(R) is zero when the force is zero. Combining equations (17) and (18) gives

$$\frac{mu^2}{2} = \frac{n+1}{2} \times E(R) \tag{19}$$

so that

Kinetic Energy × 2 = Potential Energy × ($n+1$). (20)

This tells us that for circular gravitational orbits (where n=–2), the potential energy is twice as large as the kinetic energy, and is negative. For elliptical orbits, the equation still holds: but now in terms of the *average*[3] kinetic and potential energies. Equation (20) will not hold instantaneously at all times for non-circular orbits.

1.2.3 Kepler's Laws

The motion of the planets in the Solar system was observed extensively and accurately during the Renaissance, and Kepler formulated three "laws" to describe what the astronomers saw. For the Olympiad, you won't need to be able to derive these laws from the equations of gravity, but you will need to know them, and use them (without proof).

1. All planets orbit the Sun in elliptical orbits, with the Sun at one focus.

[3] By average, we refer to the mean energy in time. In other words, if T is the orbital period, the average of A is given by $\frac{1}{T}\int_0^T A(t)dt$.

2. The area traced out by the radius of an orbit in one second is the same for a planet, whatever stage of its orbit it is in. This is another way of saying that its angular momentum is constant, and we shall be looking at this in Chapter 3.

3. The time period of the orbit is related to the [time mean] average radius of the orbit: $T \propto \langle R \rangle^{3/2}$. It is not too difficult to show that this is true for circular orbits, but it is also true for elliptic ones.

1.2.4 Large Masses

In our work so far, we have assumed that all masses are very small in comparison to the distances between them. However, this is not always the case, as you will often be working with planets, and they are large! However there are two useful facts about large spheres and spherical shells. A spherical shell is a shape, like the skin of a balloon, which is bounded by two concentric spheres of different radius.

1. The gravitational field experienced at a point outside a sphere or spherical shell is the same as if all the mass of the shape were concentrated at its centre.

2. A spherical shell has no gravitational effect on an object inside it.

These rules only hold if the sphere or shell is of uniform density (strictly – if the density has spherical symmetry).

Therefore let us work out the gravitational force experienced by a miner down a very very very deep hole, who is half way to the centre of the Earth. We can ignore the mass above him, and therefore only count the bit below him. This is half the radius of the Earth, and therefore has one eighth of its mass (assuming the Earth has uniform density – which it doesn't). Therefore the M in equation (11) has been reduced by a factor of eight. Also the miner is twice as close to the centre (R has halved), and therefore by the inverse-square law, we would expect each kilogram of Earth to attract him four times as strongly. Combining the factors of 1/8 and 4, we arrive at the conclusion that he experiences a gravitational field ½ that at the Earth's surface, that is 4.9 N/kg.

1.3 *Fluids – when things get sticky*

Questions about fluids are *really* classical mechanics questions. You can tackle them without any detailed knowledge of fluid mechanics. There are a few points you need to remember or learn, and that is what this section contains. Perfect gases are also fluids, but we will deal with them in chapter 5 – "Hot Physics".

1.3.1 Floating and ... the opposite

The most important thing to remember is Archimedes' Principle, which states that:

> *When an object is immersed in a fluid (liquid or gas), it will experience an upwards force equal to the weight of fluid displaced.*

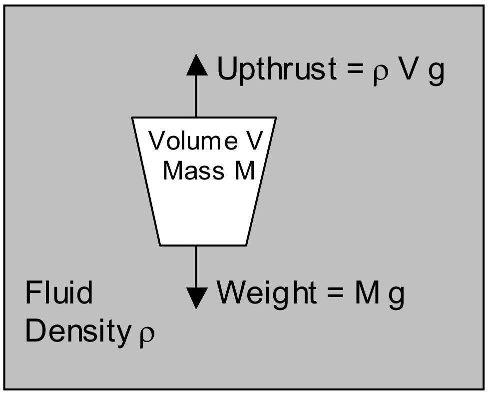
By "weight of fluid displaced" we mean the weight of the fluid that *would have been there* if the object was not in position. This upward force (sometimes called the buoyant upthrust) will be equal to

$$\begin{aligned} \text{Force} &= \text{Weight of fluid displaced} \\ &= g \times \text{Mass of fluid displaced} \\ &= g \times \text{Density of fluid} \times \text{Volume of fluid displaced} \end{aligned} \qquad (21)$$

For an object that is completely submerged, the "volume of fluid displaced" is the volume of the object.

For an object that is only partly submerged (like an iceberg or ship), the "volume of fluid displaced" is the volume of the object below the "waterline".

This allows us to find out what will float, and what will sink. If an object is completely submerged, it will have two forces acting on it. Its weight, which pulls downwards, and the buoyant upthrust, which pulls upwards.



Object floats if:

$\rho V > M$
$\rho > M/V$

Therefore, things float if their overall density (total mass / total volume) is less than the density of the fluid. Notice that the overall density may not be equal to the actual density of the material. To give an example a ship is made of metal, but contains air, and is therefore able to float because its overall density is reduced by the air, and is therefore lower than the density of water. Puncture the hull, and the air is no longer held in place. Therefore the density of the ship = the density of the steel, and the ship sinks.

For an object that is floating on the surface of a fluid (like a ship on the ocean), the upthrust and weight must be equal – otherwise it would rise or fall. From Archimedes' principle, the weight of water displaced must equal the total weight of the object.

There is a "brain-teaser" question like this: A boat is floating in the middle of a lake, and the amount of water in the lake is fixed. The boat is carrying a large rock. The rock is lifted out of the boat, and dropped into the lake. Will the level of water in the lake go up or down?

Answer: Level goes down – while the rock was in the boat (and therefore floating) its *weight* of water was being displaced. When it was dropped into the depths, its *volume* of water was displaced. Now the density of rock is higher than that of water, so the water level in the lake was higher in the first case.

1.3.2 Under Pressure

What is the pressure in a fluid? This must depend on how deep you are, because the deeper you are, the greater weight of fluid you are supporting. We can think of the pressure (=Force/Area) as the weight of a square prism of fluid above a horizontal square metre marked out in the depths.

$$\begin{aligned}\text{Pressure} &= \text{Weight of fluid over } 1\text{m}^2 \text{ square} \\ &= g \times \text{Density} \times \text{Volume of fluid over } 1\text{m}^2 \\ &= g \times \text{Density} \times \text{Depth} \times \text{Cross sectional area of fluid } (1\text{m}^2)\end{aligned}$$

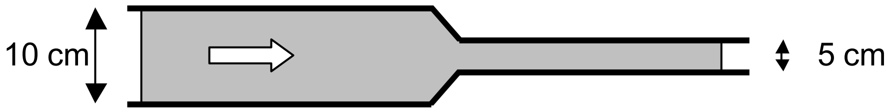
$$\text{Pressure} = g \times \text{Density} \times \text{Depth} \qquad (22)$$

Of course, this equation assumes that there is nothing pushing down on the surface of the liquid! If there is, then this must be added in too. Therefore pressure 10m under the surface of the sea = atmospheric pressure + weight of a 10m high column of water.

It is wise to take a bit of caution, though, since pressures are often given relative to atmospheric pressure (i.e. 2MPa more than atmospheric) – and you need to keep your wits about you to spot whether pressures are relative (vacuum = -100 kPa) or absolute (vacuum = 0 Pa).

1.3.3 Continuity

Continuity means conservatism! Some things just don't change – like energy, momentum, and amount of stuff. This gives us a useful tool. Think about the diagram below, which shows water in a 10cm [diameter] pipe being forced into a 5cm pipe.



Water, like most liquids, doesn't compress much – so it can't form bottlenecks. The rate of water flow (cubic metres per second) in the big pipe must therefore be equal to the rate of water flow in the little pipe.

You might like to draw an analogy with the current in a series circuit. The light bulb has greater resistance than the wire but the current in both is the same, because the one feeds the other.

How can we express this mathematically? Let us assume that the pipe has a cross sectional area A, and the water is going at speed u m/s. How much water passes a point in 1 second? Let us put a marker in the water, which moves along with it. In one second it moves u metres. Therefore volume of water passing a point = volume of cylinder of length u and cross sectional area $A = u\,A$. Therefore

$$\text{Flow rate } (\text{m}^3/\text{s}) = \text{Speed } (\text{m/s}) \times \text{Cross sectional area } (\text{m}^2). \quad (23)$$

Now we can go back to our original problem. The flow rate in both wide and narrow pipes must be the same. So if the larger one has twice the diameter, it has four times the cross sectional area; and so its water must be travelling *four times more slowly*.

1.3.4 Bernoulli's Equation

Something odd is going on in that pipe. As the water squeezes into the smaller radius, it speeds up. That means that its kinetic energy is increasing. Where is it getting the energy from? The answer is that it can only do so if the pressure in the narrower pipe is lower than in the wider pipe. That way there is an unbalanced force on the fluid in the cone-shaped part speeding it up. Let's follow a cubic metre of water through the system to work out how far the pressure drops.

The fluid in the larger pipe pushes the fluid in the cone to the right. The force = pressure × area = $P_L\,A_L$. A cubic metre of fluid occupies length $1/A_L$ in the pipe, where A_L is the cross sectional area of the pipe to the left of the constriction. Accordingly, the work done by the fluid in the wider pipe on the fluid in the cone in pushing the cubic metre through is Force × Distance = $P_L\,A_L \times 1/A_L = P_L$. However this cubic metre does work $P_R\,A_R \times 1/A_R = P_R$ in getting out the other side. Thus the net energy gain of the cubic metre is $P_L - P_R$, and this must equal the change in the cubic metre's kinetic energy $\rho u_R^2/2 - \rho u_L^2/2$.

1.3.5 The Flow Equation

Equation (23) is also useful in the context of electric currents, and can be adapted into the so-called *flow equation*. Let us suppose that the fluid contains charged particles. Suppose that there are N of these particles per cubic metre of fluid, and each particle has a charge of q coulombs, then:

$$\begin{aligned} \text{Current} &= \text{Flow rate of charge (charge / second)} \\ &= \text{Charge per cubic metre (C/m}^3) \times \text{flow rate (m}^3\text{/s)} \\ &= N\,q \times \text{Area} \times \text{Speed}\,. \end{aligned} \tag{24}$$

Among other things, this equation shows why the free electrons in a semiconducting material travel faster than those in a metal. If the semiconductor is in series with the metal, the current in both must be the same. However, the free charge density N is much smaller in the semiconductor, so the speed must be greater to compensate.

1.4 *Questions*

1. Calculate the work done in pedalling a bicycle 300m up a road inclined at 5° to the horizontal.

2. Calculate the power of engine when a locomotive pulls a train of 200 000kg up a 2° incline at a speed of 30m/s. Ignore the friction in the bearings. +

3. A trolley can move up and down a track. It's potential energy is given by $V = k\,x^4$, where x is the distance of the trolley from the centre of the track. Derive an expression for the force exerted on the trolley at any point. +

4. A ball bearing rests on a ramp fixed to the top of a car which is accelerating horizontally. The position of the ball bearing relative to the ramp is used as a measure of the acceleration of the car. Show that if the acceleration is to be proportional to the horizontal distance moved by the ball (measured relative to the ramp), then the ramp must be curved upwards in the shape of a parabola. ++

5. Use arguments similar to equation (3) to prove that the kinetic energy is still given by $\frac{1}{2}mu^2$ even when the force which has caused the acceleration from rest has not been applied uniformly in a constant direction. +

6. Calculate the final velocity of a rocket 60% of whose launch mass is propellant, where the exhaust velocity is 2000m/s. Repeat the calculation for a rocket where the propellant makes up 90% of the launch mass. In both cases neglect gravity.

7. Repeat question 6, now assuming that rockets need to move vertically in a uniform gravitational field of 9.8N/kg. Calculate the velocity at MECO (main engine cut-off) and the greatest height reached. Assume that both rockets have a mass of 10 000kg on the launch pad, and that the propellant is consumed evenly over one minute. ++

8. A 70kg woman stands on a set of bathroom scales in an elevator. Calculate the reading on the scales when the elevator starts accelerating upwards at $2m/s^2$, when the elevator is going up at a steady speed, and when the elevator decelerates at $2m/s^2$ before coming to a halt at the top floor of the building.

9. The woman in q8 is a juggler. Describe how she might have to adjust her throwing techniques in the elevator as it accelerates and decelerates.

10. Architectural models can not be properly tested for strength because they appear to be stronger than the real thing. To see why, consider a half-scale model of a building made out of the same materials. The weight is 1/8 of the real building, but the columns are ¼ the cross sectional area. Accordingly the stress on the columns is half of that in the full size building, and accordingly the model can withstand much more severe load before collapsing. To correct for this, a 1:300 architectural model is put on the end of a centrifuge arm of radius 10m which is spun around. The spinning 'simulates' an increased gravitational force which allows the model to be accurately tested. How many times will the centrifuge go round each minute?

11. Consider an incompressible fluid flowing from a 15cm diameter pipe into a 5cm diameter pipe. If the velocity and pressure before the constriction are 1m/s and 10 000 N/m^2, calculate the velocity and pressure in the constricted pipe. Neglect the effects of viscosity and turbulent flow. To work out the new pressure, remember that the increase in speed involves an increase of kinetic energy, and this energy must come from somewhere – so there will be a drop in pressure.

12. Calculate the orbital time period T of a satellite skimming the surface of a planet with radius R and made of a material with density ρ. Calculate the orbital speed for an astronaut skimming the surface of a comet with a 10km radius.

13. The alcohol percentage in wine can be determined from its density. A very light glass test tube (of cross sectional area $0.5cm^2$) has 5g of lead pellets fixed to the bottom. You place the tube in the wine, lead first, and it floats with the open end of the tube above the surface of the wine. You can read the % alcohol from markings on the side of the tube. Calculate how far above the lead the 0% and 100% marks should be placed. The density of water is $1.00g/cm^3$, while that of ethanol is

$1.98 g/cm^3$. Where should the 50% line go? Remember that alcohol percentages are always volume percentages. +