BAAO 2020/21 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

Q1 – Mercury Rotation Period with Aricebo [25 marks]

a. Calculate the power of each echo received by the Aricebo telescope and hence determine the total number of photons in each echo, given the echo was detected 579.3 s after being transmitted and Mercury's surface only reflects 6.5% of the incident radio photons. Assume θ = 0.16° and the reflected photons from Mercury are scattered randomly within only the hemisphere facing Earth.

We can use the light travel time to get the distance to Mercury

$$d_{Merc} = c \times \frac{t_{Merc}}{2} = 3.00 \times 10^8 \times \frac{579.3}{2} = 8.69 \times 10^{10} \text{ m} \quad [1]$$

From this we can work out the radius of the beam

$$r_{beam} = d_{Merc} \tan\theta = 8.69 \times 10^{10} \times \tan 0.16° = 2.43 \times 10^8 \text{ m} \quad [1]$$

[Accept $r_{beam} = d_{Merc} \tan\theta + R_{Aricebo}$ although the effect of $R_{Aricebo}$ is negligible as Mercury is so far away. Also accept small angle approximation if θ is correctly converted into radians]

Consequently the intensity of the beam at Mercury is then

$$b_{Merc} = \frac{P}{\pi r_{beam}^2} = \frac{2.0\times10^6}{\pi(2.43\times10^8)^2} = 1.08 \times 10^{-11} \text{ W m}^{-2} \quad [1]$$

This means the total power reflected is

$$P_{refl} = b_{Merc} \times \pi R_{Merc}^2 \times 0.065$$
$$= 1.08 \times 10^{-11} \times \pi(2440 \times 10^3)^2 \times 0.065 = 13.1 \text{ W} \quad [1]$$

Treating it as a point source radiating in just the hemisphere that faces Earth

$$b_{Earth} = \frac{P_{refl}}{2\pi d_{Merc}^2} = \frac{13.1}{2\pi(8.69\times10^{10})^2} = 2.77 \times 10^{-22} \text{ W m}^{-2} \quad [1]$$

Consequently, the power received by Aricebo is

$$P_{received} = b_{Earth} \times \pi R_{Aricebo}^2$$
$$= 2.77 \times 10^{-22} \times \pi \left(\frac{304.8}{2}\right)^2 = \boxed{2.02 \times 10^{-17} \text{ W}} \quad [1] \quad [6]$$

Turning this into a number of photons:

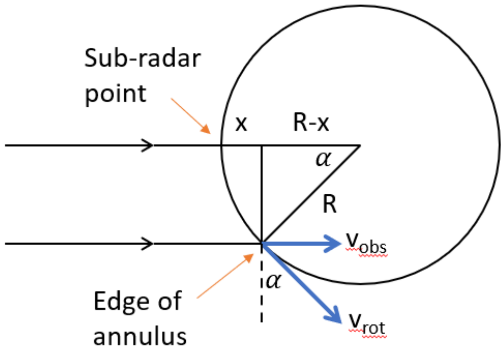
$$E_{phot} = hf = 6.63 \times 10^{-34} \times 430 \times 10^6 = 2.85 \times 10^{-25} \text{ J} \quad [1]$$

$$n = \frac{E_{tot}}{E_{phot}} = \frac{P_{recieved} t_{pulse}}{E_{phot}} = \frac{2.02\times10^{-17}\times500\times10^{-6}}{2.85\times10^{-25}} = \boxed{35500 \text{ photons}} \quad [1] \quad [2]$$

[This drop in power of 10^{23} is why you need such a powerful transmitter and such a large receiving dish to pick up the echo, and hence why it was such a hard measurement to make at the time]

b. Averaging over a series of pulses from August 1965, after correcting for the relative motion of the Earth and Mercury and the rotation rate of the Earth during the observations, the difference between the frequencies of photons from the extreme left and right parts of an annulus received 500 μs after the initial echo was Δf_{total} = 4.27 Hz.

i. Given that the pulse was Doppler shifted twice (once when reflected, once again when received back at Aricebo) show that the rotational period of Mercury is ≈ 60 days. Assume that the axis of rotation is normal to the plane of observations. [1 day = 24 hours]



Helpful diagram of the situation [1]

Calculating x from the path difference between a beam that reflects of the sub-radar point and the edge of the annulus:

$$x = c \times \frac{1}{2}t = 3.00 \times 10^8 \times \frac{1}{2} \times 500 \times 10^{-6} = 7.50 \times 10^4 \text{ m} \quad [1]$$

We can now calculate α,

$$\cos\alpha = \frac{R-x}{R} = \frac{2440-75}{2440} = 0.969 \therefore \alpha = 14.2° \quad [1]$$

Using the Doppler effect equation given, with Δf halved twice (once for the two journeys, once to compare from sub-radar point to edge rather than edge to edge),

$$v_{obs} = \frac{\frac{1}{4}\Delta f_{total}}{f}c = \frac{\frac{1}{4}\times 4.27}{430\times10^6} \times 3.00 \times 10^8 = 0.745 \text{ m s}^{-1} \quad [1]$$

Considering the geometry of the situation to get the tangential (rotational) velocity,

$$v_{rot} = \frac{v_{obs}}{\sin\alpha} = \frac{0.745}{\sin 14.2°} = 3.027 \text{ m s}^{-1} \quad [1]$$

$$\therefore T_{rot} = \frac{2\pi R_{Merc}}{v_{rot}} = \frac{2\pi\times2440\times10^3}{3.027} = 5.06 \times 10^6\ s = \boxed{58.6 \text{ days}} \quad [1] \quad [6]$$

[Students using $v_{rot} = \frac{\Delta f_{total}}{f}c = 2.98\ m\ s^{-1}$ to get $T_{rot} = 59.56$ days only get the final mark for ecf on their v_{rot} (i.e. 1 out of 6). T_{rot} can be given in seconds or days, but if in the former a comparison to 60 days must be made]

[We are fortunate that the rotational axis of mercury is almost perfectly normal to the plane of observations, so this is almost identical to the real rotational period]

ii. Mercury has a semi-major axis of 0.387 au. Rounding slightly if necessary, express the ratio orbital period : rotational period in a simple integer form (the integers should be < 10).

From Kepler's third law,

$$T_{orb} = \sqrt{\frac{4\pi^2 a^3}{GM_\odot}} = \sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times1.99\times10^{30}} \times (0.387 \times 1.50 \times 10^{11})^3} \quad [1]$$

$$= 7.63 \times 10^6 \text{ s} = 88.28 \text{ days} \quad [1]$$

We can now work out the ratio

$$\therefore \text{ratio} = \frac{T_{orb}}{T_{rot}} = \frac{88.2836}{58.617} = 1.506 \approx 1.5$$ [allow ecf with T_{rot} = 60 days] [0.5]

$$\therefore \boxed{3:2}$$ [0.5] [3]

c. The eccentricity of the planet's orbit became the prime suspect as to why its actual ratio would be stable over long time periods.

i. Calculate how many times larger the tidal torque is when Mercury is at perihelion than when it is at aphelion, given the eccentricity of the orbit is 0.206.

Given $r_{peri} = a(1-e)$ and $r_{aph} = a(1+e)$ and $\tau \propto r^{-6}$

$$\therefore \frac{\tau_{peri}}{\tau_{aph}} = \left(\frac{r_{peri}}{r_{aph}}\right)^{-6} = \left(\frac{1-e}{1+e}\right)^{-6} = \left(\frac{1-0.206}{1+0.206}\right)^{-6} = \boxed{12.3}$$ [1] [1]

ii. Assuming the tidal torque at perihelion is the dominating factor in setting Mercury's rotation rate, predict the rotational period of Mercury if it were to behave as though it was tidally locked when passing through perihelion. Compare this to the measured value and comment on validity of the assumption.

Finding the perihelion distance first

$$r_{peri} = a(1-e)$$
$$= (0.387 \times 1.50 \times 10^{11})(1 - 0.206) = 4.61 \times 10^{10}\ \text{m}$$ [1]

[Give this mark if they have already calculated r_{peri} in c) i) instead]

Using the vis-viva equation given at the beginning of the paper,

$$v_{peri} = \sqrt{GM_{\odot}\left(\frac{2}{r_{peri}} - \frac{1}{a}\right)}$$
$$= \sqrt{6.67 \times 10^{-11} \times 1.99 \times 10^{30}\left(\frac{2}{4.61 \times 10^{10}} - \frac{1}{0.387 \times 1.50 \times 10^{11}}\right)}$$
$$= 5.89 \times 10^{4}\ \text{m s}^{-1}$$ [1]

Working out the orbital angular velocity,

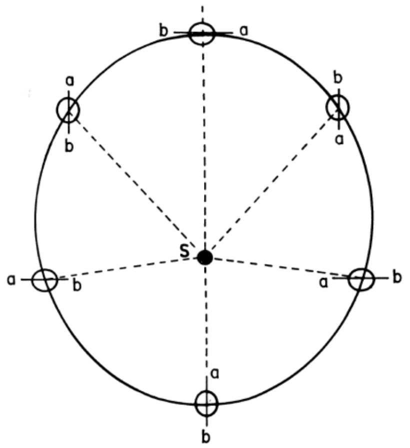
$$\omega_{orb} = \frac{v_{peri}}{r_{peri}} = \frac{5.89 \times 10^{4}}{4.61 \times 10^{10}} = 1.28 \times 10^{-6}\ \text{rad s}^{-1}$$ [1]

If $\omega_{orb} = \omega_{rot}$ [which gives $v_{rot} = 3.12\ \text{m s}^{-1}$]

$$\therefore T_{rot} = \frac{2\pi}{\omega_{rot}} = \frac{2\pi}{1.28 \times 10^{-6}} = 4.91 \times 10^{6}\ \text{s} = \boxed{56.9\ \text{days}}$$ [1] [4]

This is similar to the measured value of T_{rot} ∴ valid assumption [1] [1]

iii. Fig 3 shows the orientation of Mercury's axis of minimum moment of inertia (the axis the tidal torque acts upon) if the ratio had been 1 : 1. Redraw this diagram but for the ratio found in part b (ii). Assume the initial orientation at perihelion looks the same in both cases (so only the other five positions are needed, separated equally in time) and that the planet both orbits and rotates in an anticlockwise direction. [Note: the orientation when it returns to perihelion may not be the same as it was initially.]



New axes are alternating vertical and horizontal lines [1] [1]

Ends for a & b correctly labelled [1] [1]

[This does mean that it needs two Mercurian years before the perihelion orientation is the same – as such one (solar) day on Mercury lasts two years!]

Q2 – Great Conjunctions and the Star of Bethlehem [35 marks]

a. In ideal observing conditions the two planets are far enough apart that they should be (just about) distinguishable to the naked eye, however to some observers in imperfect conditions they would appear as a single bright dot, brighter than either planet on its own.

i. During the conjunction, the apparent magnitudes of Jupiter and Saturn were $m_J = -1.97$ and $m_S = 0.63$, respectively (ignoring dimming by the atmosphere). What would be the apparent magnitude of the two planets if they appeared to an observer as a single point? [Hint: it is not simply $-1.97 - 0.63 = -2.60$]

Given that brightness adds linearly, given the brightness of Jupiter (b_J) and of Saturn (b_S), the joint brightness of Jupiter and Saturn b_{J+S} is

$$\frac{b_{J+S}}{b_S} = \frac{b_J + b_S}{b_S} = \frac{b_J}{b_S} + 1 \quad [1]$$

Using the formula connecting magnitudes and brightness given at the start of the paper,

$$\frac{b_{J+S}}{b_S} = 10^{-0.4(m_{J+S} - m_S)} \therefore m_{J+S} = m_S - 2.5\log\left(\frac{b_J}{b_S} + 1\right)$$

$$= m_S - 2.5\log\left(10^{-0.4(m_J - m_S)} + 1\right) \quad [1]$$

$$= 0.63 - 2.5\log\left(10^{-0.4(-1.97-0.63)} + 1\right)$$

$$= \boxed{-2.06} \quad [1] \quad [3]$$

[Coincidentally, $-\sqrt{(-1.97)^2 + (0.63)^2} = -2.068$; this approach receives 0 marks]

ii. Although they appeared close in angle, there was a very considerable distance between the two planets. At conjunction, Jupiter was 5.926 au from Earth whilst Saturn was 10.827 au (see Fig 5). If they were actually next to each other in space such that they could be treated as a single object, how far from the Earth (in au) would they need to be to have the same apparent magnitude as calculated in the previous part? For simplicity, assume that both planets can be modelled as (very low luminosity) stars so that the change in brightness is only due to changing the distance from the Earth (i.e. ignore the complications from the changing distance from the Sun affecting the number of reflected photons and the changing geometry affecting the illuminated fraction of the planet's surface).

To balance the magnitudes for being at the same distance, we can use absolute magnitudes given the formula at the beginning of the paper (with distances converted into pc)

$$\mathcal{M}_J = m_J - 5\log\left(\frac{d_J}{10}\right) = -1.97 - 5\log\left(\frac{5.926 \times 1.50 \times 10^{11}}{10 \times 3.09 \times 10^{16}}\right) = 25.7$$

Similarly, $\mathcal{M}_S = 27.0$ [need both absolute magnitudes for the mark] [1]

Using a similar approach to the previous part of the question to get the combined absolute magnitude,

$$\mathcal{M}_{J+S} = \mathcal{M}_S - 2.5\log\left(10^{-0.4(\mathcal{M}_J - \mathcal{M}_S)} + 1\right)$$

$$= 27.0 - 2.5\log\left(10^{-0.4(25.7-27.0)} + 1\right) = 25.4 \quad [1]$$

$$\therefore d_{J+S} = 10\ \text{pc} \times 10^{(m_{J+S} - \mathcal{M}_{J+S})/5}$$

$$= 10\ \text{pc} \times 10^{(-2.06-25.4)/5}$$

$$= 3.15 \times 10^{-15}\ \text{pc} = 9.72 \times 10^{11}\ \text{m} = \boxed{6.48\ \text{au}} \quad [1] \quad [3]$$

[Must be in au for final mark. Make some allowance for high sensitivity to rounding errors]

[Even at a standardised distance, Jupiter is still considerably brighter. If we had kept it as reflected photons and thus had to take into account the change in incident radiation, we would get $d_{J+S} = 6.91$ au, showing that our simplifying approximation is reasonable]

b. Jupiter has a period of 4332.589 days and Saturn has a period of 10759.22 days (where 1 day = 24 hours). Note: be careful as your calculations will be very sensitive to rounding errors.

i. Calculate the time between great conjunctions as viewed from the centre of the Solar System (this will be equal to the average synodic period). Give your answer in years (where 1 year = 365.25 days). [Hint: consider a reference frame rotating at the same rate as Jupiter.]

Moving into a frame of reference where Jupiter is stationary,

$$\omega_{J+S} = \omega_J - \omega_S \therefore \frac{1}{T_{J+S}} = \frac{1}{T_J} - \frac{1}{T_S} \quad [1]$$

$$\therefore T_{J+S} = \left(\frac{1}{4332.589} - \frac{1}{10759.22}\right)^{-1} = 7253.455 \text{ days} = \boxed{19.86 \text{ years}} \quad [1] \quad [2]$$

[Must be in years for the final mark. First mark is for any sensible approach in terms of ω]

ii. Use your answer to predict the date (to the nearest day) of the next great conjunction.

Rewriting T_{J+S} into something more convenient

$$T_{J+S} = 20 \text{ years} - 51.545 \text{ days}$$

∴ next conjunction 52 days <u>before</u> 21st Dec [1]

52 days = 21 in Dec + 30 in Nov + 1 in Oct ∴ $\boxed{30^{\text{th}} \text{ Oct } 2040}$ [2] [3]

[One mark for the year (2040), one for the date. Allow ±1 day. Alternatively, some students may look for the 304th day in 2040]

[The real date is 31st Oct 2040 so this model has done remarkably well – due to the dependence on the Earth's position in its orbit the gap between conjunctions can vary up to ~ 320 days above or below the average value]

iii. Some astronomers have suggested that the 'Star of Bethlehem' seen by the magi ('wise men') on their way to Jesus' birth was in fact a great conjunction. Use your average synodic period to find the date of the great conjunction in the first decade BC and give your answer to the nearest month. [Note: be careful with BC years as year 0 in your calculation is equivalent to 1 BC, since 31st December 1 BC is followed by 1st January 1 AD]

In 2020 years there have been $\frac{2020}{19.86} = 101.7$ cycles

∴ need 102 cycles before 2020 conjunction [1]

Start date = 10 days before the end of the year $= 2020 + \frac{356}{365.25} = 2020.975$ [1]

[Give this mark if they have already calculated a decimalised start date in b) ii) instead. The top of the fraction must be 356 as 2020 is a leap year – lose this mark for other values]

∴ end date $= 2020.975 - (102 \times 19.86) = -4.63$ [1]

0.63 years = 7.57 months before 31st Dec ∴ May [1]

Since a number between 0 and 1 corresponds to 1 BC, between -1 and 0 corresponds to 2BC, and so on, ∴ a number between -5 and -4 corresponds to 6 BC ∴ $\boxed{\text{May 6 BC}}$ [1] [5]

[One mark for the year (6BC), one for the month. Allow ±1 month]

[From our data a more precise prediction is 230.4 days before 31st Dec and so 14th May 6 BC. The real great conjunction in that decade occurred on 3rd June 6 BC, so is rather close to our predicted date. This is before the death of King Herod in 4 BC, as required to fit in with the story, but it would have been seen in the constellation of Pisces in the NNE, so is unlikely to have been a 'star in the East']

c. By empirically fitting a sinusoidal function (which is assumed to be the same for each track, just with a fixed phase difference between them) and assuming all conjunctions are separated by the average synodic period, we can give rough estimations for the separations of any given great conjunction. Note: be careful as your calculations will be very sensitive to rounding errors.

i. By reading off the graph, give an equation for Track A of the form $\theta = \left|D \sin\left(\frac{2\pi t}{\lambda} + \phi_A\right)\right|$, where t is the (decimalised) date in years, and D, λ, and $-\pi/2 < \phi_A \leq \pi/2$ are values that need to be determined. [Hint: ensure your function passes through the 2020 data point, and the function is decreasing as it does.]

Reading off the graph the amplitude (D) and the period (λ) of the sine wave:

$D = 1.20°$ [allow $1.10° \leq D \leq 1.30°$] [1] [1]

$\lambda = 2500$ years [allow ± 300 years] [1] [1]

[Throughout the rest of this question, propagate the student's values accordingly]

[Note: in reality D is set by the difference in the inclinations of Jupiter and Saturn's orbits]

For ϕ_A the required properties are:

Correct behaviour (i.e. function is decreasing as it passes through 2020) [1]

In the specified range (i.e. $-\pi/2 < \phi_A \leq \pi/2$) [1] [2]

Guidance on awarding the marks for ϕ_A:

$$\phi_A = \sin^{-1}\left(\frac{0.102}{1.2}\right) - \frac{2\pi \times 2020.975}{2500} = -4.99 \text{ rad } (= -286.1°)$$

Not in right range, or right behaviour so 0 marks

$$\phi_A = -4.99 + \pi = -1.85 \text{ rad } (= -106.1°)$$

Not in right range, or right behaviour so 0 marks

$$\phi_A = -4.99 + 2\pi = 1.29 \text{ rad } (= 73.8°)$$

In right range, but not right behaviour so 1 mark

$$\phi_A = \left(\pi - \sin^{-1}\left(\frac{0.102}{1.2}\right)\right) - \frac{2\pi \times 2020.975}{2500} = -2.02 \text{ rad } (= -115.9°)$$

Not in right range, but right behaviour so 1 mark

$$\phi_A = -2.02 + \pi = \boxed{1.12 \text{ rad}} \ (= 64.1°)$$

In right range and right behaviour so 2 marks

[So final formula is $\theta = \left|1.20 \sin\left(\frac{2\pi t}{2500} + 1.12\right)\right|$, but do not penalise students for not writing this so long as their values of D, λ and ϕ_A are clear]

ii. Without having to read anything else off the graph, write down the equations for Tracks B and C, given the same restrictions on ϕ_B and ϕ_C.

D and λ remain the same [this can be implied by calculations in later parts] [1] [1]

The phase difference is $\pi/3$, the only extra consideration being staying in the required range

$$\phi_B = \phi_A - \frac{2\pi}{3} = \boxed{-0.98 \text{ rad}} \ (= -55.9°)$$ [1] [1]

$$\theta = \left|1.20 \sin\left(\frac{2\pi t}{2500} - 0.98\right)\right|$$

$$\phi_C = \phi_A - \frac{\pi}{3} = \boxed{0.07 \text{ rad}} \ (= 4.1°)$$ [1] [1]

$$\theta = \left|1.20 \sin\left(\frac{2\pi t}{2500} + 0.07\right)\right|$$

[If they go outside the required range e.g. $\phi_B = \phi_A + \frac{\pi}{3} = 2.17 \text{ rad } (= 124.1°)$ then lose that mark, unless they have already been penalised in part i. for being outside the range (in which case allow full ecf) . As in the previous part, do not penalise students for not writing the equations in the form $\theta = \cdots$ so long as their values of D, λ and ϕ_A are clear]

iii. State which track the 'Star of Bethlehem' great conjunction is on, and hence use the relevant equation to predict its separation. How does it compare to the 2020 great conjunction?

Want 102 cycles before 2020 great conjunction, and 102 is divisible by 3

$\therefore$ Track A [1] [1]

$$\therefore \theta = \left|1.20 \sin\left(\frac{2\pi t}{2500} + 1.12\right)\right|$$

$$= \left|1.20 \sin\left(\frac{2\pi \times -4.63}{2500} + 1.12\right)\right| = \boxed{1.07°}$$ [1] [1]

This is much further apart than the 2020 great conjunction [1] [1]

[Note: other tracks coincidentally give similar values of θ; award 0 marks if wrong track used]

[Since this separation is over 10 times larger than the 2020 great conjunction (the real separation was almost the same as in this model), this adds evidence to suggest it would not have been as spectacular in the sky (and certainly not close enough to look like a single star), so that particular great conjunction is probably not an explanation for the Star of Bethlehem]

iv. When is the next great conjunction at least as close as the 2020 one (i.e. $\theta \leq 0.102°$)? Give its year and the value of θ.

Rather than using brute force to find it, the graph suggests a good one to try is the next point after 2020 on Track A

$\therefore t = 2020.975 + (3 \times 19.86) = 2080.55$ $\therefore$ year $= \boxed{2080}$ [1] [1]

$\therefore \theta = \left|1.20 \sin\left(\frac{2\pi \times 2080.55}{2500} + 1.12\right)\right| = \boxed{0.078°}$ (so good guess) [1] [1]

[Using intuition here saved us a lot of time. The real date is 15th March 2080, and the real separation will actually be almost identical to the 2020 great conjunction]

v. Similarly, when was the last great conjunction at least as close as the 2020 one? Give its year and the value of θ.

Again, to avoid brute force, the graph suggests we see whether there is anything suitable around Track B's last x-intecept

$$0 = 1.20 \sin\left(\frac{2\pi t}{2500} - 0.98\right) \therefore t = \frac{2500 \times 0.98}{2\pi} = 388$$

These repeat every ½ λ so the last one was $388 + \frac{2500}{2} = 1638$ [1]

Looking at the number of track intervals you need to go back,

$n = \frac{(2020.975 + 19.86) - 1638}{3 \times 19.86} = 6.76$ $\therefore$ try 7 intervals back from 2040 [1]

$\therefore t = (2020.975 + 19.86) - (7 \times 3 \times 19.86) = 1623.80$

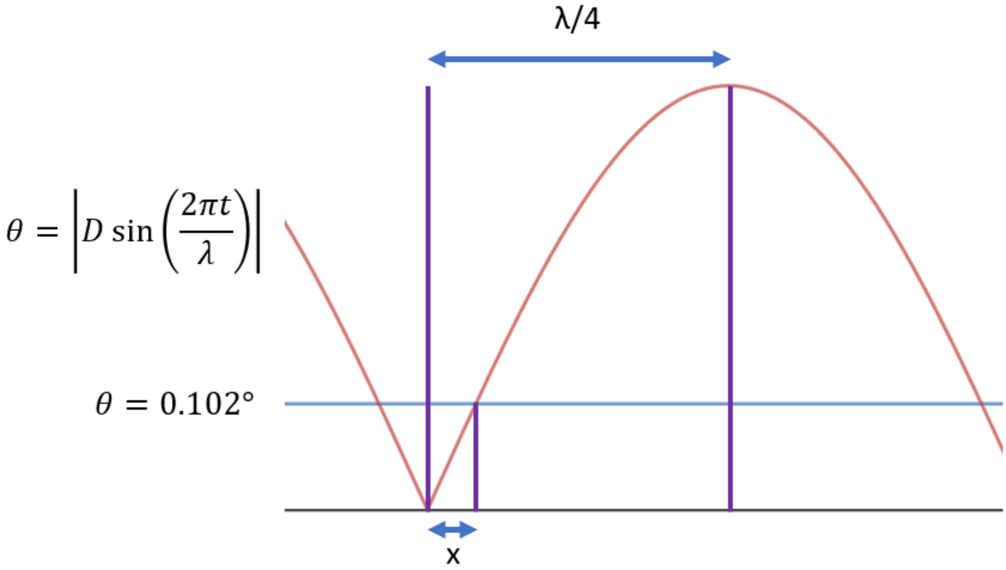
$\therefore$ year $= \boxed{1623}$ [1] [3]

$\therefore \theta = \left|1.20 \sin\left(\frac{2\pi \times (2020.975 + 19.86)}{2500} - 0.98\right)\right| = \boxed{0.043°}$ (good guess) [1] [1]

[Another example of intuition saving us lots of time. We have added the 19.86 to the 2020.975 to move from Track A to Track B. The real one was on 16th July 1623 (our model predicts 19th October) and had a separation of 0.086°. Some closer separations do happen, including transits (where the disc of Jupiter overlaps with the disc of Saturn), the next of which are due to happen in 7541 and 8674, and occultations (where Jupiter completely blocks out Saturn) in 7541, 13340 and 13738. (Note: 7541 sees part of a sequence called a triple conjunction, where retrograde motion means that Jupiter and Saturn meet three times in a short period. The transit is in February and the occultation is in June)]

vi. Using your equations, calculate the probability that a great conjunction has $\theta \leq 0.102°$.

We need to know for how long in each cycle the curve has a value below 0.102° and compare that to the length of each cycle to get the overall probability



Using the notation of the diagram above,

$$\text{probability} = \frac{x}{\lambda/4} \qquad [1]$$

We can ignore phase information as that will not affect the size of x, just the co-ordinates of its start and end points, and hence we can simplify the expression for θ to find x

$$0.102 = 1.20 \sin\left(\frac{2\pi x}{2500}\right) \therefore x = \frac{2500}{2\pi} \times \sin^{-1}\left(\frac{0.102}{1.20}\right) = 33.86 \text{ years} \qquad [1]$$

$$\therefore \text{prob} = \frac{33.86}{2500/4} = 0.0542 = \boxed{5.42\%} \qquad [1] \qquad [3]$$

[Accept it given as either a decimal or a percentage. Note that λ cancels in the expression for probability so it is only dependent on their value of D, giving prob $= \frac{\sin^{-1}(0.102/D)}{\pi/2}$; this approach gains all the marks and gives values ranging from 5.00% to 5.91% in the allowed range of D. Allow any other sensible approaches that give results within this range]

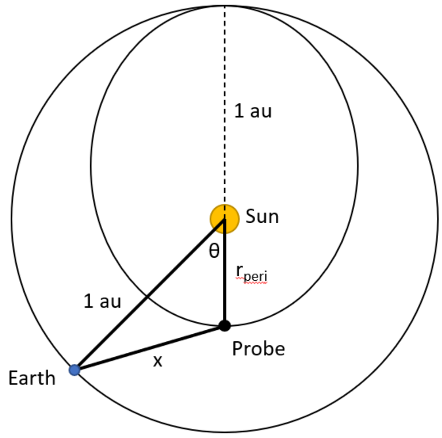
[The real value (taken from the number of conjunctions satisfying this condition over 16000 years of conjunctions) is 5.34% (38 instances out of 711) showing how well this model does. The orbits of the planets change slightly over time due to their gravitational interactions, so extending this pattern arbitrarily far into the future or past is unlikely to give precise results]

Q3 – High Resolution Stellar Photos [40 marks]

a. When it reached first perihelion, radio signals from the probe took 446.58 s to reach Earth.

i. Show that the spacecraft's perihelion is ≈ 0.5 au, giving your answer to 4 s.f., and hence estimate the launch date, assuming the Earth's orbit is circular. Note that 2020 is a leap year and take 1 year = 365.25 days. [Hint: You may wish to use a numerical method.]

This is a diagram of the situation (note: Earth and probe are orbiting anticlockwise)



Given the light travel time,

$$x = 446.58 \times 3.00 \times 10^8 = 1.3397 \times 10^{11}\text{ m} = 0.8932\text{ au} \qquad [1]$$

Considering the geometry of the triangle Earth-Sun-Probe after travelling for time t:

$$x^2 = r_{peri}^2 + (1\text{ au})^2 - 2r_{peri}(1\text{ au})\cos\theta \qquad \text{[cosine rule]} \qquad [0.5]$$

$$\theta = \pi - \frac{2\pi t}{1\text{ year}} \qquad [0.5]$$

Keeping r_{peri} in au, and given t is half the period of the probe's orbit, T, we can use Kepler's third law [= K3] to work out t in SI units:

$$a = \frac{1}{2}(1 + r_{peri})(1\text{ au}) \text{ and } (2t)^2 = \frac{4\pi^2}{GM_\odot}a^3$$

$$\therefore t = \sqrt{\frac{\pi^2}{8GM_\odot}\left((1.50 \times 10^{11})(1 + r_{peri})\right)^3} \qquad [2]$$

Substituting this into our equations above to form one where r_{peri} is the only unknown:

$$r_{peri}^2 + 1 - 2r_{peri}\cos\left(\pi - 2\pi\frac{\sqrt{\frac{\pi^2}{8GM_\odot}\left((1.50\times10^{11})(1+r_{peri})\right)^3}}{365.25\times24\times60\times60}\right) - x^2 = 0 \quad [1]$$

Using an appropriate numerical method, $r_{peri} = \boxed{0.4956\text{ au}}$ [2] [7]

[Allow any valid method to find the root of the equation such as Newton-Raphson starting with r_{peri} = 0.5 au, using a graphical calculator to find the roots, or even using trial and error (starting from 0.5 au). The final answer must be 4 s.f. otherwise a mark is dropped. If given in metres, a comparison must be made to 0.5 au (also in metres) to show they are approximately the same]

[If students use the 'simple' version of K3, $T^2 = a^3$, where T is already in years and a is already in au, then they will get $t = \sqrt{\frac{1}{32}(1 + r_{peri})^3}$ so the equation to be solved is $r_{peri}^2 + 1 - 2r_{peri}\cos\left(\pi - 2\pi\sqrt{\frac{1}{32}(1 + r_{peri})^3}\right) - x^2 = 0$, which gives $r_{peri} = \boxed{0.4998\text{ au}}$. This approach receives full marks (and is somewhat simpler to type into a calculator)]

Given we know r_{peri}, we can find a, t and therefore the launch date:

$$a = \frac{r_{peri}+1}{2} = \frac{0.4956+1}{2} = 0.7478 \text{ au} \quad [1]$$

[This mark may be given in part (iii) instead if not awarded here]

$$\therefore t = \frac{1}{2}\sqrt{\frac{4\pi^2}{GM_\odot}a^3} = \frac{1}{2}\sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times1.99\times10^{30}}(0.7478\times1.50\times10^{11})^3} \quad [1]$$

$$= 1.0244\times10^7 \text{ s} = 118.567 \text{ days} \quad [1]$$

So we need 15th June – 118.567 days

118.567 days = 15 in June + 31 in May + 30 in April + 31 in March + 11.567 in Feb

so will be 29 – 11.567 = 17.433 ∴ $\boxed{17^{\text{th}}\text{ Feb}}$ [allow 18th Feb] [1] [4]

[Using the 'simple' version of K3 gives a = 0.7499 au, t = 118.592 days, and the same launch date. This is effectively the same as if students use the r_{peri} = 0.5 au given, although doing that with the SI version of K3 gives t = 119.090 days and so 16th Feb. Using the given r_{peri} and the simple version of K3 gives t = 118.618 days. A launch date of 17th Feb corresponds to arriving at perihelion at 00:00 on 15th June, whilst 18th Feb allows for students presuming a later time of arrival (up to 23:59 on 15th June). Allow ecf on the launch date from their own values of t]

[In reality, the probe reached a perihelion distance of 0.514 au and was launched on 10th Feb. The reason for the discrepancy is that the Earth's orbit is elliptical and reaches perihelion on 4th Jan so the probe was not actually at aphelion at launch (although it did reach it on 17th Feb), and the aphelion distance was 0.986 au]

ii. Use the Ramanujan approximation to work out the distance travelled by the probe between launch and perihelion to 4 s.f.

Using the equations given at the beginning of the paper,

$$r_{peri} = a(1-e) \therefore e = 1 - \frac{r_{peri}}{a} = 1 - \frac{0.4956}{0.7478} = 0.3372 \quad [1]$$

$$e = \sqrt{1-\frac{b^2}{a^2}} \therefore b = a\sqrt{1-e^2} = 0.4956\sqrt{1-0.3372^2} = 0.7040 \text{ au} \quad [1]$$

$$\therefore j = \frac{a-b}{a+b} = \frac{0.7478-0.7040}{0.7478+0.7040} = 0.030176 \quad [1]$$

$$\therefore C = \pi(a+b)\left[1+\frac{3j^2}{10+\sqrt{4-3j^2}}\right]$$

$$= \pi(0.7478+0.7040)\left[1+\frac{3(0.030176)^2}{10+\sqrt{4-3(0.030176)^2}}\right] = 4.562 \text{ au} \quad [0.5]$$

But only half an ellipse has been covered, so distance travelled = $\boxed{2.281 \text{ au}}$ [0.5] [4]

[Must be 4 s.f. for the final mark. Carrying forward values from the 'simple' version of K3 gives e = 0.3335, b = 0.7070 au, j = 0.02947, C = 4.578 au and so distance travelled is 2.289 au (essentially the same as if using the given r_{peri})]

iii. Find the values of D and E, given as fractions in their simplest terms, and hence calculate a new value for the distance travelled by the probe (also to 4 s.f.). Compare this to the approximation in the previous part and comment on your answer.

Using the given formula for a binomial expansion,

$$I = \int_0^{\pi/2} (1 - e^2 \sin^2 t)^{1/2}\, \mathrm{d}t$$

$$= \int_0^{\pi/2} 1 + \frac{1}{2}(-e^2 \sin^2 t) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(-e^2 \sin^2 t)^2 + \cdots \mathrm{d}t$$

$$= \int_0^{\pi/2} 1 - \frac{1}{2}e^2 \sin^2 t - \frac{1}{8}e^4 \sin^4 t\, \mathrm{d}t$$ [correctly simplified expansion] [1]

$$= \int_0^{\pi/2} 1\, \mathrm{d}t - \frac{1}{2}e^2 \int_0^{\pi/2} \sin^2 t\, \mathrm{d}t - \frac{1}{8}e^4 \int_0^{\pi/2} \sin^4 t\, \mathrm{d}t$$

The first term is very easy to evaluate (and is effectively already given),

$$\int_0^{\pi/2} 1\, \mathrm{d}t = [t]_0^{\pi/2} = \frac{\pi}{2}$$

To do the other integrals we need to use the trig identities:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \text{ and } \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

The second term,

$$\int_0^{\pi/2} \sin^2 t\, \mathrm{d}t = \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) \mathrm{d}t = \left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi/2} = \frac{\pi}{4}$$ [1]

The third term,

$$\int_0^{\pi/2} \sin^4 t\, dt$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)^2 dt$$

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{4}\cos^2 2t\, dt$$

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4t\right) dt$$

$$= \int_0^{\pi/2} \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t\, dt$$

$$= \left[\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t\right]_0^{\pi/2} = \frac{3\pi}{16}$$ [3]

So putting back into our original formulae,

$$C = 4aI = 4a\left(\frac{\pi}{2} - \frac{1}{2}e^2\left[\frac{\pi}{4}\right] - \frac{1}{8}e^4\left[\frac{3\pi}{16}\right] - \cdots\right) = 2\pi a\left[1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \cdots\right]$$

$$\therefore \boxed{D = \frac{1}{4}} \text{ and } \boxed{E = \frac{3}{64}}$$ [one mark for each correct fraction] [2] [7]

We can now evaluate C numerically,

$$C \approx 2\pi a\left[1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6\right]$$

$$= 2\pi(0.7478)\left[1 - \frac{1}{4}(0.3372)^2 - \frac{3}{64}(0.3372)^4 - \frac{5}{256}(0.3372)^6\right]$$

$$= 4.562 \text{ au}$$ [0.5]

$$\therefore \text{distance travelled} = \boxed{2.281 \text{ au}}$$ [0.5] [1]

To 4 s.f., this is the same as the Ramanujan approximation (so very good!) [1] [1]

[Must be 4 s.f. for the final mark. If students forget the e^6 term, they will still get the same answer to 4 s.f. – do not penalise them for that mistake. Using the 'simple' version of K3 gives 2.289 au, again the same as before. The real difference between the two values is 4.5×10^{-6} au $= 680$ km, which is remarkably close]

[In reality, the distance the probe had travelled from launch to perihelion was 2.393 au, however the main reason for the discrepancy is the extra week of travel between launch and aphelion]

b. The energy density of black-body radiation, u, and number density, n, at temperature T are given.

i. The average energy per photon is given as $\bar{E} = \frac{u}{n} = \varepsilon k_B T$. Find the numerical value of ε.

To get the integral to look like the standard one, we need to use a substitution:

$$\text{Use } x = \frac{h\nu}{k_B T} \therefore \nu = \frac{k_B T}{h} x \therefore \mathrm{d}\nu = \frac{k_B T}{h} \mathrm{d}x$$ [need correct dv for mark] [1]

Substituting this into the original integral,

$$u = \int_0^\infty \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu = \int_0^\infty \frac{8\pi h}{c^3} \left(\frac{k_B T}{h}\right)^3 \frac{x^3}{\exp(x) - 1} \frac{k_B T}{h} dx$$ [1]

$$= \frac{8\pi h}{c^3} \left(\frac{k_B T}{h}\right)^4 \left[\frac{\pi^4}{15}\right] = \frac{8}{15} \pi^5 \left(\frac{k_B T}{hc}\right)^3 k_B T$$ [1]

$$\therefore \bar{E} = \frac{u}{n} = \frac{\frac{8}{15}\pi^5 \left(\frac{k_B T}{hc}\right)^3 k_B T}{16\pi \left(\frac{k_B T}{hc}\right)^3 \zeta(3)} = \frac{\pi^4}{30\zeta(3)} k_B T$$ [1]

$$\therefore \varepsilon = \frac{\pi^4}{30\zeta(3)} = \boxed{2.701}$$ [1] [5]

[Third mark is for some attempt at simplification of the algebraic expression. No s.f. requirement in this question but expect at least 2 s.f. (and should really use 4 s.f. in the next part of the question). If get dv = dx and hence a value of ε that still has a value of T in it this is considered a major error and there is no ecf]

ii. Assuming the plasma of Fe ions is in thermal equilibrium with the photons, and that the average energy of the photons is equal to the ionisation energy of FeX (which is 22 540 kJ mol^{-1}), calculate the temperature of the plasma. Give your answer to 4 s.f.

Converting the ionisation energy into an energy of each photon

$$\bar{E} = \frac{22540 \times 10^3}{N_A} = \frac{22540 \times 10^3}{6.02 \times 10^{23}} = 3.744 \times 10^{-17}\ \mathrm{J}$$ [1]

$$\therefore T = \frac{\bar{E}}{\varepsilon k_B} = \frac{3.744 \times 10^{-17}}{2.701 \times 1.38 \times 10^{-23}} = \boxed{1.004 \times 10^6\ \mathrm{K}}$$ [must be 4 s.f.] [1] [2]

[Despite the considerable simplifications we have made to our model, this is remarkably close to the real value of temperature for which Fe X is the dominant fraction. In reality, it forms an equilibrium over a range of temperatures (contributing a detectable fraction from about $10^{5.4}$ to $10^{6.4}$ K) with various other ions in ionisation equilibrium – all you can compute is the ionic fraction at a given temperature (i.e. the plasma is never purely one ion)]

c. The Rayleigh criterion and speed of sound in a plasma are given.

i. Determine the theoretical minimum angular diameter of an element resolvable by this optical system. Give your answer in arcseconds (").

First, we can work out the wavelength given the photon energy,

$$E = \frac{hc}{\lambda} \therefore \lambda = \frac{hc}{E} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{71.0372 \times 1.60 \times 10^{-19}} = 1.75 \times 10^{-8}\ \mathrm{m}$$ [1]

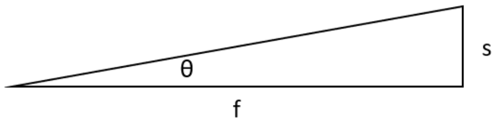
$$\therefore \theta = 1.22 \frac{\lambda}{D} = 1.22 \times \frac{1.75 \times 10^{-8}}{47.4 \times 10^{-3}} = 4.50 \times 10^{-7}\ \mathrm{rad} = \boxed{0.0929''}$$ [1] [2]

[Must be in arcseconds for the final mark]

[Given the wavelength investigated is 17.5 nm, it is clearly in the extreme UV]

ii. In practice, this is not achieved as the pixels are not small enough. Given that each picture element is spread across two pixels (in 1D) to allow adequate sampling, what is the actual minimum angle resolved on the CCD? Give your answer in arcseconds (“). [Hint: consider the geometry of the optical system and note that the angles are small enough that the small angle approximation can be used.]

Considering the geometry of the situation:



where s is the distance on the CCD and f is the focal length. Using the small angle approx:

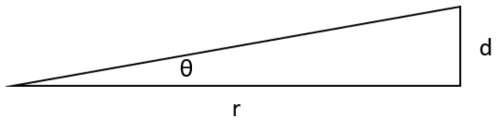
$$\tan\theta \approx \theta = \frac{s}{f} \qquad [1]$$

$$\therefore \theta = \frac{2\times10^{-6}}{4187\times10^{-3}} = 4.78\times10^{-6}\ \text{rad} = \boxed{0.985''} \qquad [1] \quad [2]$$

[Must be in arcseconds for the final mark. If only done for s = 1 pixel (i.e. 10 µm), lose 1 mark. Another valid approach for the first mark is to use the formula for the plate scale, $\frac{\theta}{s} = \frac{1}{f} = 0.239\ \text{rad m}^{-1} = 2.39\times10^{-6}\ \text{rad pixel}^{-1} = 0.493''\ \text{pixel}^{-1}$]

iii. Later on in its mission, Solar Orbiter will have a perihelion of 0.284 au. Calculate the physical size on the Sun (in km) of each picture element in an image taken with HRI_{EUV} as well as the FOV in units of $R_{\odot}$.

Using similar geometry to the previous part of the question,



where d is the distance on the Sun and r is the distance to the Sun. Using the small angle approximation again,

$$d \approx r\theta = (0.284\times1.50\times10^{11})\times4.78\times10^{-6} = \boxed{203\ \text{km}} \qquad [1] \quad [1]$$

Considering the field of view (FOV) of the telescope, we need to convert 1000” into rad

$$FOV = r\theta = (0.284\times1.50\times10^{11})\times\left(\frac{1000}{3600}\times\frac{\pi}{180}\right) = 2.07\times10^{8}\ \text{m}$$

$$= \boxed{0.297\ R_{\odot}} \qquad [1] \quad [1]$$

[So the field of view is 0.297 $R_{\odot}$ by 0.297 $R_{\odot}$. First answer must be in km and second in $R_{\odot}$ for the marks]

[Originally the mission plan was to go even closer, however it was adjusted to have a minimum perihelion of > 0.28 au so that the solar panel technology could be reused from the BepiColombo mission, which will enter an orbit around Mercury in December 2025. Mercury’s perihelion is ≈ 0.30 au so Solar Orbiter will pass fully inside the orbit of Mercury. The minimum perihelion for Solar Orbiter will be on 7th Feb 2027]

iv. If the mass fractions of the surface of the Sun are X = 0.7381, Y = 0.2485, and Z = 0.0134, and treating the plasma as an ideal monatomic gas so that $\gamma = 5/3$, determine the speed of sound of the plasma (in km s^{-1}) at the base of the corona. Hence, by comparison to your answer from the previous part, estimate the upper limit on the length of an exposure to avoid motion blur in the plasma. You should ignore any motion blur from the relative motion of the spacecraft or the rotation of the Sun.

Using the given mass fractions and formulae, plus the temperature from b. (ii)

$$\mu = \frac{m_p}{2X+\frac{3Y}{4}+\frac{Z}{2}} = \frac{m_p}{2(0.7381)+\frac{3(0.2485)}{4}+\frac{0.0134}{2}} = 0.599m_p = 1.00 \times 10^{-27}\ \text{kg} \quad [1]$$

$$\therefore v_s = \sqrt{\frac{\gamma k_B T}{\mu}} = \sqrt{\frac{\frac{5}{3}\times 1.38\times 10^{-23}\times 1.004\times 10^{6}}{1.00\times 10^{-27}}} = \boxed{152\ \text{km s}^{-1}} \quad [1] \quad [2]$$

[Must be in km s^{-1} for the final mark. If students did not get a value for T from part b. they should have seen in the opening paragraph of the question that typical corona temperatures are 10^6 K and used that – penalise those that use the temperate of the surface of the Sun]

By comparing to the physical size of each picture element,

$$t = \frac{d}{v_s} = \frac{203}{152} = \boxed{1.34\ \text{s}} \quad [1] \quad [1]$$

[The HRI_{EUV} will take images of 1 second exposure (with shorter exposure times in subfields of the FOV) every second in the 'Discovery' program, however most images will have a 2 second exposure with the interval time between images varying from seconds to minutes depending on the different observation programs]