BAAO 2018/19 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. If a candidate gets the final (numerical) answer then allow them all the marks for that part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made.

Q1 – Parker Solar Probe [25 marks]

a. *When the probe is at its closest perihelion:*

i. *Calculate the apparent magnitude of the Sun, given that from Earth $m_\odot$ = -26.74.*

$$\text{ratio of brightnesses, } \frac{\mathrm{b}_1}{\mathrm{b}_0} = \frac{(1\ \mathrm{au})^2}{(9.86\ \mathrm{R}_\odot)^2} = \frac{(1.50\times10^{11})^2}{(9.86\times6.96\times10^8)^2} = 478 \qquad [1]$$

$$\therefore m_{\mathrm{new}} = m_\odot - 2.5\log\left(\frac{\mathrm{b}_1}{\mathrm{b}_0}\right) = -26.74 - 6.698$$

$$= -33.44 \qquad [1] \quad [2]$$

[Allow calculating the solar absolute magnitude, $\mathcal{M}_\odot$ = 4.829, as an alternative first mark]

ii. *Calculate the temperature the heat shield must be able to survive. Assume that the heat shield of the probe absorbs all of the incident radiation, radiates as a perfect black body, and that only one side of the probe ever faces the Sun (to protect the instruments) such that the emitting (surface) area is double the absorbing (cross-sectional) area.*

For thermal balance, power absorbed must equal power emitted, so

$$\frac{L_\odot}{4\pi r^2} \times A_{abs} = \sigma A_{emit} T^4$$

But since $A_{\mathrm{emit}} = 2 \times A_{\mathrm{abs}}$ then

$$\frac{L_\odot}{4\pi r^2} = 2\sigma T^4 \quad \therefore \quad T = \sqrt[4]{\frac{L_\odot}{8\pi\sigma r^2}} \qquad [1]$$

$$= \sqrt[4]{\frac{3.85\times10^{26}}{8\pi\times5.67\times10^{-8}\times(9.86\times6.96\times10^8)^2}} \qquad [1]$$

$$= 1550\ \mathrm{K} \qquad [1] \quad [3]$$

b. *Given that in its final orbit PSP has an orbital period of 88 days, calculate the speed of the probe as it passes through the minimum perihelion. Give your answer in km s^{-1}.*

Using Kepler's third law to find the semi-major axis of the final orbit,

$$T^2 = \frac{4\pi^2}{GM}a^3 \quad \therefore \quad a = \sqrt[3]{\frac{GM}{4\pi^2}T^2}$$

$$= \sqrt[3]{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{4\pi^2}\times(88\times24\times3600)^2} \qquad [1]$$

$$= 5.79\times10^{10}\ \mathrm{m} \qquad [1]$$

$$v = \sqrt{GM\left(\frac{2}{r}-\frac{1}{a}\right)} = \sqrt{6.67\times10^{-11}\times1.99\times10^{30}\left(\frac{2}{9.86\times6.96\times10^8}-\frac{1}{5.79\times10^{10}}\right)} \qquad [1]$$

$$(= 190766\ \mathrm{m\,s^{-1}}) = 191\ \mathrm{km\,s^{-1}} \qquad \text{[Must be in km s}^{-1}\text{]} \qquad [1] \quad [4]$$

[This will mean that the Parker Solar Probe will become the fastest spacecraft (relative to the Sun) ever flown – at this speed you could travel from New York to Tokyo in less than a minute!]

c. *After the first flyby of Venus on 3rd October 2018 it was moved into an orbit with a 150 day period, and the subsequent first perihelion on 6th November 2018 was at a distance of 35.7 $R_\odot$. Given its mass at launch was 685 kg, calculate the total amount of energy that had to be lost by the probe to get from this first orbit (ignoring the orbital properties prior to the Venus flyby) to the final orbit. Ignore any change in the mass of the probe due to burning fuel.*

Using a similar method to the previous question to work out the semi-major axis and perihelion speed for the first orbit,

$$a_{initial} = 8.27 \times 10^{10}\ \text{m} \qquad [1]$$
$$v_{initial} = 95.3\ \text{km s}^{-1} \qquad [1]$$

The total energy is the sum of the potential and kinetic energies,

$$E_{tot} = E_P + E_K \ \text{ and } E_P = -\frac{GMm}{r}$$

At the first perihelion,

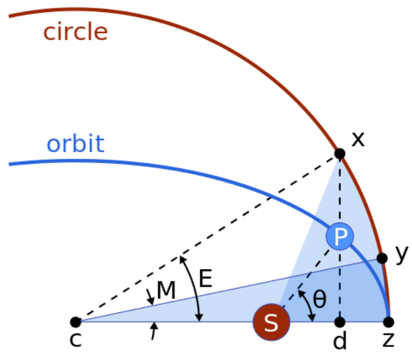
$$E_{tot} = -\frac{6.67\times10^{-11}\times1.99\times10^{30}\times685}{35.7\times6.96\times10^{8}} + \frac{1}{2}\times 685 \times (95279)^2$$
$$= -3.66\times10^{12} + 3.11\times10^{12}$$
$$= -5.50\times10^{11}\ \text{J}$$ [Allow 1 mark if only have a correct E_P or E_K term] [2]

At the final perihelion,

$$E_{tot} = -\frac{6.67\times10^{-11}\times1.99\times10^{30}\times685}{9.86\times6.96\times10^{8}} + \frac{1}{2}\times 685 \times (190766)^2$$
$$= -1.32\times10^{13} + 1.25\times10^{13}$$
$$= -7.85\times10^{11}\ \text{J}$$ [Allow 1 mark if only have a correct E_P or E_K term] [2]

$$\therefore \Delta E_{tot} = 2.35\times10^{11}\ \text{J}$$ [Ignore a minus sign] [1] [7]

d. *Derive a formula for the distance from the focus for an elliptical orbit, r (SP in the figure) in terms of the semi-major axis a, the eccentricity e, and the eccentric anomaly E.*



In this diagram:
cx = **cz** = *a*
SP = *r*
cS = *ae*
cd = *U*
Pd = *V*
therefore **Sd** = **cd** – **cs** = U – ae

For an ellipse, the general equation is $\frac{U^2}{a^2} + \frac{V^2}{b^2} = 1$, so the points (U,V) on the ellipse are

$$\cos E = \frac{U}{a} \qquad \text{and} \qquad \sin E = \frac{V}{b} \qquad [1]$$

We are also given that $e = \sqrt{1 - \frac{b^2}{a^2}} \quad \therefore \quad b^2 = a^2(1-e^2)$

Using Pythagoras' theorem,

$$(\boldsymbol{SP})^2 = (\boldsymbol{Pd})^2 + (\boldsymbol{Sd})^2$$
$$\therefore\ r^2 = (b\sin E)^2 + (a\cos E - ae)^2 \qquad [1]$$
$$= a^2(1-e^2)\times(1-\cos^2 E) + a^2(\cos^2 E - 2e\cos E + e^2) \qquad [1]$$
$$= a^2 - 2a^2 e\cos E + a^2e^2\cos^2 E$$
$$= a^2(1 - e\cos E)^2$$
$$\therefore r = a(1 - e\cos E) \qquad [1] \quad [4]$$

[Third mark is for eliminating *b* and writing all expressions in terms of only one trigonometric function. A reasonable attempt at a derivation (allowing alternative methods) must be present to get the marks for this question (simply writing the answer only scores 1 mark)]

e. Calculate how long PSP spends doing primary science in its final orbit. Give your answer in days.

With the perihelion distance we can find the eccentricity,

$$r_{peri} = a(1-e) \;\therefore\; e = 1 - \frac{r_{peri}}{a} = 1 - \frac{9.86\times6.96\times10^{8}}{5.79\times10^{10}} = 0.882 \qquad [1]$$

Using the formula just derived we can find *E* for *r* = 0.25 au,

$$E = \cos^{-1}\left(\frac{1}{e}\left(1-\frac{r}{a}\right)\right) = \cos^{-1}\left(\frac{1}{0.882}\left(1-\frac{0.25\times1.50\times10^{11}}{5.79\times10^{10}}\right)\right) = 1.16\text{ rad } (= 66.4°)\ [1]$$

Using the formula given we can find M,

$$M = E - e\sin E = 1.16 - 0.882\times\sin 1.16 = 0.351\text{ rad } (= 20.1°) \qquad [1]$$

Allowing for both sides of the ellipse, $\Delta M = 0.703\text{ rad } (= 40.3°)$ [1]

Movement through the circular orbit has constant angular velocity, meaning

$$\frac{\Delta M}{\Delta t} = \frac{2\pi}{T} \;\therefore\; \Delta t = \frac{T\Delta M}{2\pi} = \frac{88\times0.703}{2\pi} = 9.84\text{ days } (= 9\text{ days } 20\text{ hours } 12\text{ mins}) \qquad [1] \qquad [5]$$

[If they forget the factor of two (so Δ*t* = 4.92 days) allow 4 marks. Allow ±1 hour on the final answer to account for intermediate rounding errors. Allow full ecf for using their value of the semi-major axis from part b. Must be given in days for the final mark]

Q2 – 360 Days in a Year [25 marks]

a. *How many sidereal days elapse during a year? Give your answer to 2 d.p.*

$$n_{sid}T_{sid} = n_{sol}T_{sol} \quad \therefore \quad n_{sid} = \frac{n_{sol}T_{sol}}{T_{sid}} = \frac{365.25\times24\times3600}{(23\times3600)+(56\times60)+4} \qquad [1]$$
$$= 366.25 \text{ (sidereal days)} \qquad [1] \quad [2]$$

[Must be 2 d.p. for the final mark]

b. *Without further calculation, suggest how many sidereal days there would be if a year was in fact only 360 solar days.*

361 (sidereal days) [1] [1]

c. *What reduction in the Earth's semi-major axis would be required for the year to be shortened down to 360 solar days?*

Using Kepler's third law to work out the new semi-major axis,

$$T^2 = \frac{4\pi^2}{GM}a^3 \qquad \therefore a_{new} = \sqrt[3]{\frac{GM}{4\pi^2}T^2}$$
$$= \sqrt[3]{\frac{6.674\times10^{-11}\times1.989\times10^{30}}{4\pi^2}\times(360\times24\times3600)^2} \qquad [1]$$
$$= 1.482\times10^{11}\ \mathrm{m} \qquad [1]$$
$$\Delta a = a_{\oplus} - a_{new} = 1.496\times10^{11} - 1.482\times10^{11}$$
$$= 1.43\times10^{9}\ \mathrm{m} \qquad [1] \quad [3]$$

[If they don't use the values of fundamental constants given in the question, resulting in a value of $\Delta a = 1.83\times10^9$ m, give 2 marks max]

d. *Imagine creating an incredibly powerful rocket, positioned on the Earth's equator, that when fired once can apply a huge force to the Earth in a very short time period, delivering a total impulse of Δp. Assuming the Earth's orbit is initially circular, calculate:*

   i. *The total impulse required to slow the Earth's rotation to give a year of 360 solar days, but with no change in the orbit.*

The sidereal day describes the actual rate of rotation of the Earth, so we need to work out the change in angular velocity going from 366.25 sidereal days to 361 sidereal days

$$\omega_{initial} = \frac{2\pi}{(23\times3600)+(56\times60)+4} = 7.2921\times10^{-5}\ \mathrm{rad\ s^{-1}} \qquad [1]$$
$$\omega_{final} = \frac{2\pi}{(24\times3600\times365.25/361)} = 7.1876\times10^{-5}\ \mathrm{rad\ s^{-1}} \qquad [1]$$
$$\therefore \Delta\omega = 1.0454\times10^{-6}\ \mathrm{rad\ s^{-1}} \qquad [1]$$

Evaluating the moment of inertia for the Earth,

$$I = \frac{2}{5}M_{\oplus}R_{\oplus}^2 = \frac{2}{5}\times5.972\times10^{24}\times(6371\times10^3)^2$$
$$= 9.6961\times10^{37}\ kg\ m^2 \qquad [1]$$
$$\Delta p = \frac{\Delta L}{R_{\oplus}} = \frac{I\Delta\omega}{R_{\oplus}} = \frac{9.6961\times10^{37}\times1.0454\times10^{-6}}{6371\times10^3} \qquad [1]$$
$$= 1.59\times10^{25}\ \mathrm{kg\ m\ s^{-1}} \qquad [1] \quad [6]$$

[If students use solar days rather than sidereal days giving $\Delta\omega = 1.0453\times10^{-6}$ rad s^{-1} then give 5 marks max (though it will give the same Δp to 4 s.f.). If they don't use the values of fundamental constants given in the question, don't penalise them in this part]

ii. *The total impulse required to change the orbit to give a year of 360 solar days, but with no change in the length of a solar day, also explaining how the rocket needs to be fired.*

We need to change the orbital speed of the Earth so that that point becomes an aphelion in an elliptical orbit with the same semi-major axis as calculated in part c.

$$v_{initial} = \frac{2\pi \times 1\text{ au}}{1\text{ year}} = \frac{2\pi \times 1.496\times10^{11}}{24\times3600\times365.25} = 29786\text{ m s}^{-1} \qquad [1]$$

$$v_{final} = \sqrt{GM_\odot\left(\frac{2}{1\text{ au}} - \frac{1}{a_{new}}\right)}$$

$$= \sqrt{6.674\times10^{-11}\times1.989\times10^{30}\left(\frac{2}{1.496\times10^{11}} - \frac{1}{1.482\times10^{11}}\right)} \qquad [1]$$

$$= 29644\text{ m s}^{-1} \qquad [1]$$

$$\therefore \Delta v = 141.4\text{ m s}^{-1} \qquad [1]$$

$$\therefore \Delta p = M_\oplus \Delta v = 5.972\times10^{24}\times141.4 \qquad [1]$$

$$= 8.44\times10^{26}\text{ kg m s}^{-1} \qquad [1] \quad [6]$$

[If they don't use the values of fundamental constants given in the question, resulting in a value of Δp = 1.81 × 10^{27} kg m s^{-1}, give 5 marks max for this calculation. Allow a tolerance of ±0.20 × 10^{26} kg m s^{-1} to account for intermediate rounding errors. Give full ecf on value for a_{new} taken from part c.]

The rocket needs to be fired towards the horizon at sunrise [1] [1]

[Accept a clear diagram showing the reaction force of the rocket on the ground creating a clockwise moment in the orbit and pointed at the centre of the Earth. Allow sunset if it's clearly consistent with the direction of rotation of the Earth on its axis and as it moves around the Sun, since they weren't explicitly specified as anti-clockwise in the question]

e. *Tidal interactions between the Moon and the Earth mean that the Earth's rotation rate is slowing down, such that a solar day has lengthened over the last 2800 years by an average of 2.3 ms per century. Similar interactions between the Earth and the Sun, as well as mass loss by the Sun due to nuclear fusion and the solar wind, mean that the distance between them is increasing by about 1.5 cm per year. Assuming these rates have stayed constant over time and that the Earth's orbit has remained circular throughout, is there any time in either the Earth's past or future when it had or when it will have a year with 360 solar days? Give your answer in Myr (where 1 Myr = 10^6 years). [For reference, the age of the Earth is 4543 Myr.]*

In 1 Myr, $a_\oplus$ has increased by 15 km and T_{sol} has lengthened by 23 s [need both] [1]

We want to satisfy the condition

$$\frac{\text{new year duration}}{\text{new solar day}} = 360 \qquad [1]$$

Using Kepler's third law for the new year duration,

$$\frac{\sqrt{\frac{4\pi^2}{6.674\times10^{-11}\times1{,}989\times10^{30}}(1.496\times10^{11}+15000x)^3}}{(24\times3600)+23x} = 360 \qquad [1]$$

$x = 54.5$ Myr (positive value indicates it's in the future) [3] [6]

[Allow any valid method to solve the cubic, such as using their graphical calculator to plot the graph and find roots, or even a trial and error iteration. Some will use a first order binomial expansion to expand the brackets and get a linear expression leading to 54.88 Myr – give full marks for this approach. Using less precise fundamental constants gives a value of 69.98 Myr (and 70.63 Myr for the linear approach) – give 5 marks max. If the student suggests it is in the past give 5 marks max]

Q3 – Stellar Mass Limits [25 marks]

a. *Given the Sun's composition has hydrogen fraction, X = 0.72, helium fraction Y = 0.26 and 'metals' (i.e. any element lithium and heavier) fraction Z = 0.02, estimate the temperature at the centre of the Sun.*

$$\bar{\mu} = \frac{m_p}{2X+3Y/4+Z/2} = \frac{1.67\times10^{-27}}{(2\times0.72)+(3\times0.26/4)+0.02/2} = 1.02\times10^{-27}\ \mathrm{kg}\ (= 0.608\ m_p) \quad [1]$$

$$T_{int} = \frac{GM_{\odot}\bar{\mu}}{k_B R_{\odot}} = \frac{6.67\times10^{-11}\times1.99\times10^{30}\times1.02\times10^{27}}{1.38\times10^{-23}\times6.96\times10^{8}} = 1.40\times10^{7}\ \mathrm{K} \quad [1] \quad [2]$$

b. *Classically, two protons need to have enough energy to overcome their electrostatic repulsion in order to fuse. Calculate the value of $T_{classical}$ necessary to allow fusion to occur, given that at that point b = 1 fm (= 10^{-15} m). [You should find that it's much larger than your answer to part a.]*

Assuming both protons are heading straight towards each other,

$$\text{Total } E_K = 2\times\frac{3}{2}k_B T_{classical} \quad [1]$$

Equating with the potential energy at the point where they can fuse,

$$\text{Total } E_K = E_P \therefore T_{classical} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{b}\frac{1}{2\times\frac{3}{2}k_B} \quad [1]$$

$$= \frac{1}{4\pi\times8.85\times10^{-12}}\frac{(1.60\times10^{-19})^2}{10^{-15}}\frac{1}{3\times1.38\times10^{-23}} \quad [1]$$

$$= 5.56\times10^{9}\ \mathrm{K} \quad [1] \quad [4]$$

[Assuming one proton is stationary, resulting in $T_{classical}$ = 1.11 × 10^{10} K, gives 3 marks max]

c. *Given that the proton momentum is related to the average kinetic energy of a particle in the plasma by $E_K = p^2/2m_p$ calculate the value of λ and hence calculate $T_{quantum}$. [You should find that it's below your answer to part a.]*

$$\lambda = \frac{h}{p} \therefore p = \frac{h}{\lambda} \quad \text{and} \quad 2\times E_K = 2\times\frac{p^2}{2m_p} = E_P = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\lambda}$$

$$\therefore \frac{h^2}{\lambda^2 m_p} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\lambda} \quad \therefore \quad \lambda = \frac{4\pi\epsilon_0 h^2}{m_p e^2} \quad [1]$$

$$= \frac{4\pi\times8.85\times10^{-12}\times(6.63\times10^{-34})^2}{1.67\times10^{-27}\times(1.60\times10^{-19})^2} \quad [1]$$

$$= 1.14\times10^{-12}\ \mathrm{m} \quad [1] \quad [3]$$

[First mark is for eliminating *p* and equating with the expression for E_P. Allow full ecf for this calculation if assuming the stationary proton case, leading to λ = 5.72 × 10^{-13} m]

$$\frac{3}{2}k_B T_{quantum} = \frac{h^2}{2\lambda^2 m_p} \therefore T_{quantum} = \frac{h^2}{3\lambda^2 m_p k_B} \quad [1]$$

$$= \frac{(6.63\times10^{-34})^2}{3\times(1.14\times10^{-12})^2\times1.67\times10^{-27}\times1.38\times10^{-23}} \quad [1]$$

$$= 4.86\times10^{6}\ \mathrm{K} \quad [1] \quad [3]$$

[First mark is for an expression for $T_{quantum}$ in terms of known variables. In the stationary proton case you get 1.95 × 10^7 K, which is above the value in part a. and so only 2 marks max for this calculation]

[Despite our simplifying assumptions, this is close to the real value of the temperature needed in a star like to Sun to undergo hydrogen fusion, which is ≈ 3 × 10^6 K. Deuterium fusion is possible at lower temperatures since each molecule has more mass and hence more momentum at a given temperature, so some very small stars (called brown dwarfs) achieve the required temperature of ≈ 4.5 × 10^5 K. The ability to fuse for even a short time is how large gas giants and small stars are distinguished, and is investigated further in the next part of the question]

d. *Assuming the star to be of uniform density at this limit with ρ = $m_p n_e$ and the electrons to be in thermal equilibrium with the plasma, show that the minimum mass of a star for which $T_{int} = T_{quantum}$ is ≈ 0.1 $M_\odot$.*

$$M_{min} = \frac{4}{3}\pi R^3 \rho = \frac{4}{3}\pi R^3 m_p n_e = \frac{4}{3}\pi R^3 \frac{m_p}{\lambda_e^3} \qquad \therefore R = \lambda_e \left(\frac{M_{min}}{\frac{4}{3}\pi m_p}\right)^{1/3} \qquad [1]$$

$$\frac{p_e^2}{2m_e} = \frac{3}{2}k_B T_{quantum} \therefore p_e = \sqrt{3 m_e k_B T_q} \qquad (= 1.35 \times 10^{-23} \text{ kg m s}^{-1}) \qquad [1]$$

$$\therefore \lambda_e = \frac{h}{p_e} = \frac{h}{\sqrt{3 m_e k_B T_q}} \qquad (= 4.90 \times 10^{-11} \text{ m}) \qquad [1]$$

$$T_{int} = \frac{G M_{min} \bar{\mu}}{k_B R} \therefore M_{min} = \frac{T_q k_B R}{G \bar{\mu}} = \frac{T_q k_B}{G\bar{\mu}} \frac{h}{\sqrt{3 m_e k_B T_q}} \left(\frac{M_{min}}{\frac{4}{3}\pi m_p}\right)^{1/3}$$

$$\therefore M_{min}^{2/3} = \frac{\sqrt{T_q k_B}}{G\bar{\mu}} \frac{h}{\sqrt{3 m_e}} \left(\frac{4}{3}\pi m_p\right)^{-1/3} \qquad [1]$$

$$= \frac{\sqrt{4.86\times10^6 \times 1.38\times10^{-23}}}{6.67\times10^{-11}\times1.02\times10^{-27}} \frac{6.63\times10^{-34}}{\sqrt{3\times9.11\times10^{-31}}} \left(\frac{4}{3}\pi \times 1.67 \times 10^{-27}\right)^{-1/3} \qquad [1]$$

$$= 2.54 \times 10^{19} \text{ kg}^{2/3} \qquad [1]$$

$$\therefore M_{min} = 1.28 \times 10^{29} \text{ kg} = 0.064\, M_\odot$$ [Must be in $M_\odot$] [1] [7]

[If students assume the star is a fully ionised pure hydrogen plasma, such that $\bar{\mu} = 0.5\, m_p$ and leading to M_{min} = 0.086 $M_\odot$ give full marks (this was the intended answer for this question, but the value of $\bar{\mu}$ to choose was ambiguous so most students will use their value from part a.) If they use the value of $T_{quantum}$ from the stationary proton case, getting M_{min} = 0.18 $M_\odot$, give 6 marks max]

[The real lower limit for a star to join the main sequence is about 0.08 $M_\odot$, or about 83 times the mass of Jupiter. In practice, the theoretical lower limit varies with the metallicity of the star – for a star like the Sun it is 0.075 $M_\odot$, whilst for a pure hydrogen star it is 0.092 $M_\odot$ (most stars we can see have a metallicity that falls between these two)]

e. *By balancing the radiative acceleration with the gravitational acceleration at the surface of a star, derive a formula for L_{Edd} in terms of M, and hence calculate the maximum mass of a star with a hydrogen fraction like the Sun. Give your answer in $M_\odot$.*

Balancing the radiative acceleration term with the gravitational one,

$$g_{rad} = g_{grav} \therefore \frac{\kappa_e I}{c} = \frac{GM}{R^2} \qquad [1]$$

$$I = \frac{L_{Edd}}{4\pi R^2} \qquad \therefore L_{Edd} = \frac{4\pi G M c}{\kappa_e} \quad \left(= \frac{8\pi G M c m_p}{\sigma_T (1+X)}\right) \qquad [1] \quad [2]$$

[L_{Edd} can be given in terms of κ or in terms of σ_T for the second mark]

$$\kappa_e = \frac{\sigma_T}{2 m_p}(1+X) = \frac{66.5\times10^{-30}}{2\times1.67\times10^{-27}}(1+0.72) = 0.03425 \text{ m}^2 \text{ kg}^{-1} \qquad [1]$$

Using the given mass-luminosity relation,

$$\frac{L_{Edd}}{L_\odot} = \left(\frac{M_{max}}{M_\odot}\right)^3 \therefore \frac{4\pi G M_{max} c}{L_\odot \kappa_e} = \frac{M_{max}^3}{M_\odot^3} \therefore M_{max} = \sqrt{\frac{4\pi G c M_\odot^3}{L_\odot \kappa_e}} \qquad [1]$$

$$= \sqrt{\frac{4\pi\times6.67\times10^{-11}\times3.00\times10^8\times(1.99\times10^{30})^3}{3.85\times10^{26}\times0.03425}} \qquad [1]$$

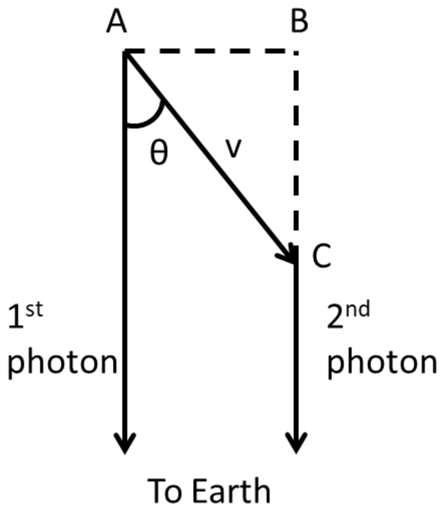
$$= 3.88 \times 10^{32} \text{ kg} = 195\, M_\odot \qquad [1] \quad [4]$$

[The answer must be in $M_\odot$ for the final mark. Full marks can be awarded without κ being calculated explicitly. Watch for incorrect conversions from fm² to m² (1 fm² = 10^{-30} m²) with σ_T]

[The real upper limit on stars is about 200 $M_\odot$, so this simplified model has done rather well]

Q4 – Superluminal Jets [25 marks]

a. *Show, with use of an appropriate diagram, that the apparent value of the scaled transverse speed (for a jet coming towards us) is* $\beta_{app} = \frac{\beta \sin\theta}{1-\beta\cos\theta}$.



Attempt at suitable diagram [be flexible] [2]

Apparent distance moved in time Δt,

$$AB = d_{app} = v\sin\theta \times \Delta t \quad [1]$$

Difference in distances moved by photons,

$$BE = c\Delta t - v\cos\theta \times \Delta t$$

So apparent time delay,

$$\Delta t_{app} = \frac{BE}{c} = \Delta t - \beta\cos\theta \times \Delta t \quad [1]$$

So apparent speed,

$$v_{app} = \frac{d_{app}}{\Delta t_{app}} = \frac{\Delta t \times v\sin\theta}{\Delta t - \beta\cos\theta \times \Delta t} \quad [1] \quad [5]$$

$$\therefore \beta_{app} = \frac{v_{app}}{c} = \frac{\beta\sin\theta}{1-\beta\cos\theta}$$

[Allow any convincing derivation, given that it was a 'show that' question. Any derivation without a diagram scores 3 marks max. Most diagrams with two sightlines, v & θ should score at least 1 mark]

b. *Determine the relationship between β and θ that maximises β_{app} for a given value of β, and hence determine the minimum value of β needed to give rise to superluminal apparent speeds (i.e. when $\beta_{app}^{max} > 1$). [You are given that a graph of β_{app} against θ has only one turning point in the range 0 < θ < π, and that it is a maximum.]*

Want to find where $\frac{d\beta_{app}}{d\theta} = 0$, so need to differentiate our expression [with product / quotient rule]

$$\frac{d\beta_{app}}{d\theta} = \frac{\beta\cos\theta(1-\beta\cos\theta) - \beta\sin\theta(\beta\sin\theta)}{(1-\beta\cos\theta)^2} = 0 \quad [1]$$

$$\therefore (\beta_{app} \text{ is maximised for a given value of } \beta \text{ when}) \ \beta = \cos\theta \quad [1] \quad [2]$$

Putting this into the formula from part a. and evaluating for when $\beta_{app} = 1$,

$$\beta_{app}^{max} = \frac{\cos\theta\sin\theta}{1-\cos\theta\cos\theta} = \cot\theta \quad \left(= \frac{\beta}{\sqrt{1-\beta^2}}\right) \quad [1]$$

$$\text{If } \cot\theta = 1, \quad \theta = \frac{\pi}{4} \quad \therefore \beta_{min} = \cos\theta = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad (= 0.707) \quad [1] \quad [2]$$

c. *Calculate β_{app} for both jets, and use your formula from part b. to calculate the minimum value of β to explain the apparent superluminal motion.*

We need to convert μ from milliarcseconds a day to radians a second,

$$v_a = \mu_a d_{system} = \frac{23.6\times10^{-3}}{3600} \times \frac{2\pi}{360} \times (11 \times 10^3 \times 3.09 \times 10^{16}) \quad [1]$$

$$= 4.50 \times 10^8 \ m\,s^{-1} \quad \therefore \beta_{app,a} = 1.50 \quad [1] \quad [2]$$

Similarly, for the receding jet, $\beta_{app,r} = 0.636$ [2] [2]

[Values of μ_a and μ_r are 1.32×10^{-12} rad s^{-1} and 5.61×10^{-13} rad s^{-1} respectively]

$$\beta_{app}^{max} = \cot\theta = 1.50 \quad \therefore \beta_{min,a} = \cos(\cot^{-1} 1.50) = 0.832 \quad [1] \quad [1]$$

[No requirement to use the data from the receding jet (which gives $\beta_{min,r} = 0.824$)]

d. Derive a formula for the distance, D, as a function of θ, μ_a and μ_r (i.e. independent of β), and hence calculate θ.

Rearranging the given formulae,

$$D\mu_a(1-\beta\cos\theta) = \beta c\sin\theta \quad \text{and} \quad D\mu_r(1+\beta\cos\theta) = \beta c\sin\theta$$

$$D\mu_a\mu_r\left(\frac{1}{\beta}-\cos\theta\right) = \mu_r c\sin\theta \ (1) \qquad D\mu_a\mu_r\left(\frac{1}{\beta}+\cos\theta\right) = \mu_a c\sin\theta \ (2) \qquad [1]$$

$$(2)-(1) \qquad D\mu_a\mu_r(2\cos\theta) = (\mu_a-\mu_r)c\sin\theta$$

$$\therefore D = \frac{c}{2}\left(\frac{\mu_a-\mu_r}{\mu_a\mu_r}\right)\tan\theta \qquad [1] \quad [2]$$

[This is just one approach – allow any reasonable attempt that gets the right answer]

$$\theta = \tan^{-1}\left(\frac{2D}{c}\left(\frac{\mu_a\mu_r}{\mu_a-\mu_r}\right)\right)$$

$$= \tan^{-1}\left(\frac{2\times 11\times 10^3\times 3.09\times 10^{16}}{3.00\times 10\text{^}8}\left(\frac{1.32\times 10^{-12}\times 5.61\times 10^{-13}}{1.32\times 10^{-12}-5.61\times 10^{-13}}\right)\right) \quad (=\tan^{-1}2.206) \qquad [1]$$

$$= 65.6° \quad (= 1.15\text{ rad}) \qquad [1] \quad [2]$$

e. Show that βcosθ can be expressed purely as a function of μ_a and μ_r, and hence use your value of θ to calculate the value of β.

Rearranging the given formulae again,

$$D\mu_a(1-\beta\cos\theta) = \beta c\sin\theta \ (1) \quad \text{and} \quad D\mu_r(1+\beta\cos\theta) = \beta c\sin\theta \ (2)$$

$$(2)-(1) \qquad D\mu_r + D\mu_r\beta\cos\theta + D\mu_a\beta\cos\theta - D\mu_a = 0 \qquad [1]$$

$$\therefore \beta\cos\theta = \frac{\mu_a-\mu_r}{\mu_a+\mu_r} \qquad [1] \quad [2]$$

[This is just one approach – allow any reasonable attempt that gets the right answer]

$$\beta\cos\theta = \frac{23.6-10.0}{23.6+10.0} = 0.405 \qquad [1]$$

$$\therefore \beta = \frac{0.405}{\cos 65.6°} = 0.981 \qquad [1] \quad [2]$$

[When they were discovered, these blobs of plasma in the jet from the microquasar were the fastest moving that anyone had ever seen in our galaxy!]

f. Greiner et. al (2001) measure K_d = 140 km s^{-1}, P_{orb} = 33.5 days, and a mass ratio for the two objects of M_{BH}/M_* = 12.3. Using the assumption that i = θ, calculate M_{BH}. Give your answer in $M_\odot$.

Rewriting the given equation in terms of the ratio M_*/M_{BH},

$$\frac{M_{BH}^3\sin^3\theta}{M_{BH}^2\left(1+\frac{M_*}{M_{BH}}\right)^2} = \frac{P_{orb}K_d^3}{2\pi G} \quad \therefore M_{BH} = \frac{P_{orb}K_d^3}{2\pi G}\frac{\left(1+\frac{M_*}{M_{BH}}\right)^2}{\sin^3\theta} \qquad [1]$$

$$= \frac{(33.5\times 24\times 3600)\times(140\times 10^3)^3}{2\pi\times 6.67\times 10^{-11}}\times\frac{(1+12.3^{-1})^2}{\sin^3 65.6°} \qquad [1]$$

$$= 2.93\times 10^{31}\text{ kg} = 14.7\,M_\odot \qquad [1] \quad [3]$$

[Must be in $M_\odot$ for the final mark]

[At the time of discovery, this meant the black hole in the microquasar was the heaviest stellar black hole known. Subsequent measurements have made the distance a little closer than 11 kpc, and so the mass has been revised down, but until the discovery of the black holes behind detections of gravitational waves (with masses about double this) it was still one of heaviest known]