BAAO 2017/18 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. If a candidate gets the final (numerical) answer then allow them all the marks for that **part** of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made.

Q1 - Stellar Mass Loss [Total = 25]

a. *Given that, at the distance of the Earth, the proton flux from the Sun's stellar wind (which is assumed to be radiated equally in all directions) is 3.0×10^{12} m^{-2} s^{-1}, and that the luminosity of the Sun is solely due to the fusion of hydrogen to helium:*

i. *Show that the rate at which the Sun is losing mass, $\dot{M} \equiv dM/dt$, due to its stellar wind is $\sim 10^{-14}\ M_{\odot}\ \mathrm{yr}^{-1}$. [Take the mass of a proton to be 1.67×10^{-27} kg.]*

$$\begin{aligned}\dot{M}_{\text{wind}} &= \text{mass flux} \times \text{surface area of sphere 1 au in radius} \\ &= (3.0 \times 10^{12} \times 1.67 \times 10^{-27}) \times (4\pi \times (1.50 \times 10^{11})^2) \quad [1] \\ &= 1.42 \times 10^{9}\ \mathrm{kg\ s^{-1}} \\ &= 2.24 \times 10^{-14}\ M_{\odot}\ \mathrm{yr^{-1}} \quad (\text{so } \sim 10^{-14}\ M_{\odot}\ \mathrm{yr^{-1}}) \quad [1] \quad [2]\end{aligned}$$

ii. *Show that $\dot{M}$ due to nuclear fusion is greater than that from the solar wind.*

$$L = \dot{M}_{\text{fusion}} c^2 \quad \therefore \quad \dot{M}_{\text{fusion}} = L_{\odot}/c^2$$

$$\begin{aligned}&= \frac{3.85 \times 10^{26}}{(3.00 \times 10^{8})^2} \quad [1] \\ &= 4.28 \times 10^{9}\ \mathrm{kg\ s^{-1}} \quad [1] \quad [2] \\ &(= 6.78 \times 10^{-14}\ M_{\odot}\ \mathrm{yr^{-1}})\ [so > \dot{M}_{\text{wind}}]\end{aligned}$$

iii. *Estimate how much the Earth-Sun distance and the Earth's orbital period will have changed after the Sun has lost mass (via both routes) for one year. Assume the orbit remains circular throughout and ignore gravitational effects from all other bodies.*

$$\dot{M}_{\text{tot}} = \dot{M}_{\text{wind}} + \dot{M}_{\text{fusion}} = 9.02 \times 10^{-14}\ M_{\odot}\ \mathrm{yr^{-1}} \quad [1]$$

In solar units, Kepler's 3rd Law is $M = a^3 / T^2$, and since the change is mass is very small compared with $M_{\odot}$ we can make the approximation $\frac{\Delta a}{a} = \frac{\Delta T}{T} = \frac{\Delta M}{M}$ [2]

So:

$$\Delta a = \frac{\Delta M}{M_{\odot}} a = 9.02 \times 10^{-14} \times 1.50 \times 10^{11} = 1.35 \times 10^{-2}\ \mathrm{m} \quad [1]$$

$$\Delta T = \frac{\Delta M}{M_{\odot}} T = 9.02 \times 10^{-14} \times (365.25 \times 24 \times 3600)$$

$$= 2.85 \times 10^{-6}\ \mathrm{s} \quad [1] \quad [5]$$

[Allow angular momentum conservation arguments too, as well as ΔT being 3 times larger]

b. *If every photon leaving a star is able to transfer all of its momentum (given by $p_{photon} = E_{photon}/c$) to drive a stellar wind (known as the single scattering limit):*

i. *Show that the maximum mass loss rate for a star can be written in terms of the luminosity of the star, L, the (terminal) velocity of the stellar wind far from the star, v_∞, and the speed of light, c, as $\dot{M}_{max} = L/v_\infty c$.*

$p_{\max} = m v_\infty \quad \therefore \quad \frac{p_{\max}}{t} = \dot{M}_{\max} v_\infty$ [1]

In the single scattering limit: $p_{\max} = p_{\text{photon}} = \frac{E}{c}$ $\quad \therefore \frac{E}{ct} = \dot{M}_{\max} v_\infty$ [1]

(since L = E/t) $\quad \therefore \frac{L}{c} = \dot{M}_{\max} v_\infty$ [1] [3]

$\left(\therefore \dot{M}_{\max} = \frac{L}{v_\infty c}\right)$

ii. *Hence, derive an expression for the maximum kinetic energy deposited per second by the stellar wind as a function of the luminosity of the star. Comment on your answer.*

$$\dot{E}_{\text{K,max}} = \frac{1}{2}\dot{M}_{max} v_\infty^2$$

$$= \frac{1}{2}\frac{L}{v_\infty c} v_\infty^2 = \frac{1}{2}\frac{v_\infty}{c} L$$ [1] [1]

Since v_∞ << c, this is << L (so photons carry away much more energy) [1] [1]

[Also allow that as $v_\infty \rightarrow c$, $E_{K,max} \rightarrow$ ½L]

c. *The Wolf-Rayet star WR7 is in the constellation of Canis Major and its strong winds are responsible for the nebula known as Thor's Helmet. The star has a mass of 16 $M_\odot$, a radius of 1.41 $R_\odot$ and a surface temperature of 112 000 K, with a measured v_∞ of 1545 km s^{-1}.*

i. *Predict the mass loss rate of WR7 based upon the properties of the star, assuming the single scattering limit. Give your answer in units of $M_\odot\, yr^{-1}$.*

$$L = 4\pi R^2 \sigma T^4$$

$$= 4\pi \times (1.41 \times 6.96 \times 10^8)^2 \times 5.67 \times 10^{-8} \times (112000)^4$$ [1]

$$= 1.08 \times 10^{32}\ \text{W} \qquad (= 2.80 \times 10^5\ L_\odot)$$ [1]

$$\dot{M}_{\max} = \frac{L}{v_\infty c} = \frac{1.08\times10^{32}}{1545\times10^3\times3.00\times10^8} = 2.33 \times 10^{17}\ \text{kg s}^{-1}$$

$$= 3.69 \times 10^{-6}\ M_\odot\ \text{yr}^{-1}$$ [1] [3]

[Award 2 max if they forget to convert the velocity into m s^{-1}]

ii. *The nebula is 5 arcmins in diameter (1° = 60 arcmin) and 4.8 kpc away, and at its edge is a bright thin shell of swept up material expanding at a rate of 30 km s^{-1}. The age of such a nebula, t, is related to the current values of radius, R, and expansion speed, v, by t = 0.55R/v. Using this model, determine the age of the nebula.*

Radius = (2.5 arcmin in radians) × (distance in metres) (we're given the diameter)

$$= \left(\frac{2.5}{60} \times \frac{\pi}{180}\right) \times (4.8 \times 10^3 \times 3.09 \times 10^{16})$$ [1]

$$= 1.08 \times 10^{17}\ \text{m} \qquad (= 3.49\ \text{pc})$$ [1]

$$t = \frac{0.55R}{v} = \frac{0.55\times1.08\times10^{17}}{30\times10^3} = 1.98 \times 10^{12}\ \text{s} \qquad (\approx 63000\ \text{years})$$ [1] [3]

[Award 2 max if they forget to convert the velocity into m s^{-1}]

iii. *If the expansion is purely driven by the direct impact of the stellar winds, then the radius at time t, R(t), can be related to $\dot{M}$ with the given formula. If n_0 = 16 cm^{-3} and m_H = 1.67 × 10^{-27} kg, calculate the observed mass loss rate based upon the properties of the nebula. Compare it with the predicted one from earlier and comment on your answer.*

$$R(t) = \left(\frac{3\dot{M}v_\infty t^2}{2\pi n_0 m_\mathrm{H}}\right)^{\frac{1}{4}} \therefore \dot{M} = \frac{2\pi n_0 m_\mathrm{H} R(t)^4}{3 v_\infty t^2}$$

$$= \frac{2\pi \times 16\times10^6 \times 1.67\times10^{-27} \times (1.08\times10^{17})^4}{3\times1545\times10^3\times(1.98\times10^{12})^2} \quad [1]$$

$$= 1.25\times10^{18}\ \mathrm{kg\ s^{-1}} \quad (= 1.99\times10^{-5}\ M_\odot\ \mathrm{yr^{-1}}) \quad [1] \quad [2]$$

This is (about 5.4 times) larger than the previously predicted value, suggesting the single scattering limit is not a good approximation in this situation [1] [1]

[In practice, multiple scattering occurs due to the Doppler Effect in the winds]

iv. *Using your new value for $\dot{M}$, calculate the total mass expelled from the star and hence the total kinetic energy the stellar wind has so far deposited into the ISM during this stage of the star's life.*

$$\text{Total mass}, M_\mathrm{tot} = \dot{M}t = 1.25\times10^{18}\times1.98\times10^{12}$$

$$= 2.48\times10^{30}\ \mathrm{kg} \quad (= 1.25\ M_\odot) \quad [1] \quad [1]$$

$$\text{Total } E_\mathrm{K,tot} = \frac{1}{2}M_\mathrm{tot}v_\infty^2 = \frac{1}{2}\times2.48\times10^{30}\times(1545\times10^3)^2$$

$$= 2.96\times10^{42}\ \mathrm{J} \quad [1] \quad [1]$$

[This means Wolf Rayet stars are depositing a very large amount of kinetic energy into the ISM in only a few tens of thousands of years (in this case about 2.5% of the total energy the Sun will radiate in its lifetime!)]

Q2 - Multi-Messenger Astronomy [Total = 25]

a. *The galaxy NGC 4993 is measured to have a redshift of z = 0.00980 ± 0.00079. Assuming it follows Hubble's Law, $v = H_0 d$, where $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}, as determined by Hubble Space Telescope (HST) measurements of Cepheid variables, calculate the distance to the galaxy (in Mpc) and its (absolute) uncertainty. Give your distance to an appropriate number of significant figures.*

$v = H_0 d \quad \therefore \; d = \frac{v}{H_0} = \frac{cz}{H_0} = \frac{3.00\times10^5\times0.00980}{73.24}$ (c must be in km s^{-1}) [1]

$= 40.1 \text{ Mpc}$ [1]

Answer given to 3 s.f. as redshift has lowest precision (penalise more s.f.) [1] [3]

Uncertainty: $\frac{\Delta d}{d} = \frac{\Delta z}{z} + \frac{\Delta H_0}{H_0} = \frac{0.00079}{0.00980} + \frac{1.74}{73.24} (= 10.4\%)$ [1]

$\therefore \Delta d = 4.2 \text{ Mpc}$ [1] [2]

Overall: $d = 40.1 \pm 4.2 \text{ Mpc}$

[Allow uncertainties added in quadrature to give 8.4%, leading to Δd = 3.4 Mpc. Also, this is the only part of the question with a s.f. penalty]

b. *For NGC 4993 we measure R_e = 15.5 arcseconds, σ = 171 km s^{-1}, and $\langle I_r \rangle_e = 407\, L_\odot$ pc^{-2}. Given that the scatter in the FP relation introduces an uncertainty in D of ±17%, calculate the distance to the galaxy (in Mpc) and its (absolute) uncertainty using the FP relation.*

$$D = \left(10^{-\log R_e + 1.24 \log\sigma - 0.82\log\langle I_r\rangle_e + 2.194}\right)(1+z)^2$$

$$= \left(10^{-\log 15.5 + 1.24\log 171 - 0.82\log 407 + 2.194}\right)(1+0.00980)^2 \quad [1]$$

$$= 43.8 \text{ Mpc} \quad [1] \quad [2]$$

$\Delta D = 17\%$ of 43.8 Mpc $= 7.4$ Mpc [1] [1]

Overall: $D = 43.8 \pm 7.4 \text{ Mpc}$

[Allow the uncertainty in z to be propagated too, giving ΔD = 14.5 Mpc or 9.0 Mpc (in quadrature)]

c. *Given that the gravitational frequency, f_{GW}, is twice the orbital frequency (i.e. $f_{GW} = \omega/\pi$) and the 'chirp mass', $\mathcal{M} = (\mu^3 M_{tot}^2)^{1/5}$, express h in terms of only $\mathcal{M}$, r, f_{GW}, and various fundamental constants.*

Using Kepler's 3rd Law: $GM_{\text{tot}} = a^3\omega^2 \quad \therefore \quad a = \left(\frac{GM_{\text{tot}}}{\omega^2}\right)^{1/3}$ [1]

Using the definition of chirp mass: $\mathcal{M} = (\mu^3 M_{\text{tot}}^2)^{\frac{1}{5}} \quad \therefore \quad \mu = \left(\frac{\mathcal{M}^5}{M_{\text{tot}}^2}\right)^{1/3}$ [1]

Putting these into the given equation: $h = \frac{G}{c^4}\frac{1}{r}\mu a^2\omega^2$

$$= \frac{G}{c^4}\frac{1}{r}\left(\frac{\mathcal{M}^5}{M_{\text{tot}}^2}\right)^{1/3}\left(\frac{GM_{\text{tot}}}{\omega^2}\right)^{2/3}\omega^2 \quad [1]$$

$$= \frac{G^{5/3}}{c^4}\frac{1}{r}\mathcal{M}^{5/3}\omega^{2/3} \quad [1]$$

Finally, using that $\omega = \pi f_{\text{GW}}$: $h = \frac{(G\mathcal{M})^{5/3}}{c^4}\frac{1}{r}\pi^{2/3}f_{\text{GW}}^{2/3}$ [1] [5]

d. *Combine your result from c. with the above equation to cancel out $\mathcal{M}$ and so express the distance to the gravitational wave source, r, as a function of fundamental constants and the measurables h, f_{GW}, and $\dot{f}_{GW}$ only.*

Given that $\dot{f}_{\rm GW} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f_{\rm GW}^{11/3}$ $\quad\therefore\quad (G\mathcal{M})^{5/3} = \frac{5}{96}\frac{c^5 \dot{f}_{\rm GW}}{\pi^{8/3} f_{\rm GW}^{11/3}}$ [1]

Using our result from part c. $\quad (G\mathcal{M})^{5/3} = \frac{hrc^4}{\pi^{2/3} f_{\rm GW}^{2/3}}$ [1]

Combining these together: $\quad \frac{5}{96}\frac{c^5 \dot{f}_{\rm GW}}{\pi^{8/3} f_{\rm GW}^{11/3}} = \frac{hrc^4}{\pi^{2/3} f_{\rm GW}^{2/3}} \quad\therefore\quad r = \frac{5}{96}\frac{c}{h\pi^2}\frac{\dot{f}_{\rm GW}}{f_{\rm GW}^3}$ [1] [3]

e. *Typically, you measure $\tau \equiv f_{GW}/\dot{f}_{GW}$, rather than $\dot{f}_{GW}$ directly. Given that just as the merger began the detectors measured $\tau = 0.0023$ s, $f_{GW} = 300$ Hz and $h = 6.0 \times 10^{-21}$, estimate the distance to GW170817 (in Mpc) and its absolute uncertainty (assuming a percentage uncertainty of ±10%). How does this compare with your answers in parts a. and b.?*

$$r = \frac{5}{96}\frac{c}{h\pi^2}\frac{1}{\tau f_{\rm GW}^2} = \frac{5}{96} \times \frac{3.00\times10^8}{6.63\times10^{-34}\times\pi^2} \times \frac{1}{0.0023\times300}$$ [1]

$$= 1.27 \times 10^{24}\ \text{m} = 41.3\ \text{Mpc}$$ [1] [2]

$$\Delta r = 10\%\ \text{of}\ 41.3\ \text{Mpc} = 4.1\ \text{Mpc}$$ [1] [1]

Overall: $\quad r = 41 \pm 4$ Mpc

The values of d, D and r are consistent with each other within their uncertainties [1] [1]

f. *Using the redshift information from NGC 4993 and the gravitational wave distance you have just calculated, determine the Hubble constant H_0 in units of km s^{-1} Mpc^{-1}, along with its absolute uncertainty. Is this value consistent with the one derived by the HST using Cepheid variables (given in part a.)?*

$v = H_0 r \quad\therefore\quad H_0 = \frac{v}{r} = \frac{cz}{r} = \frac{3.00\times10^5\times0.00980}{41.3}$ (c must be in km s^{-1}) [1]

$$= 71.3\ \text{km s}^{-1}\ \text{Mpc}^{-1}$$ [1] [2]

Uncertainty: $\quad \frac{\Delta H_0}{H_0} = \frac{\Delta z}{z} + \frac{\Delta r}{r} = \frac{0.00079}{0.00980} + 0.1 \quad (= 18.1\%)$ [1]

$$\therefore \Delta H_0 = 12.9\ \text{km s}^{-1}\ \text{Mpc}^{-1}$$ [1] [2]

Overall: $\quad H_0 = 71 \pm 13$ km s^{-1} Mpc^{-1}

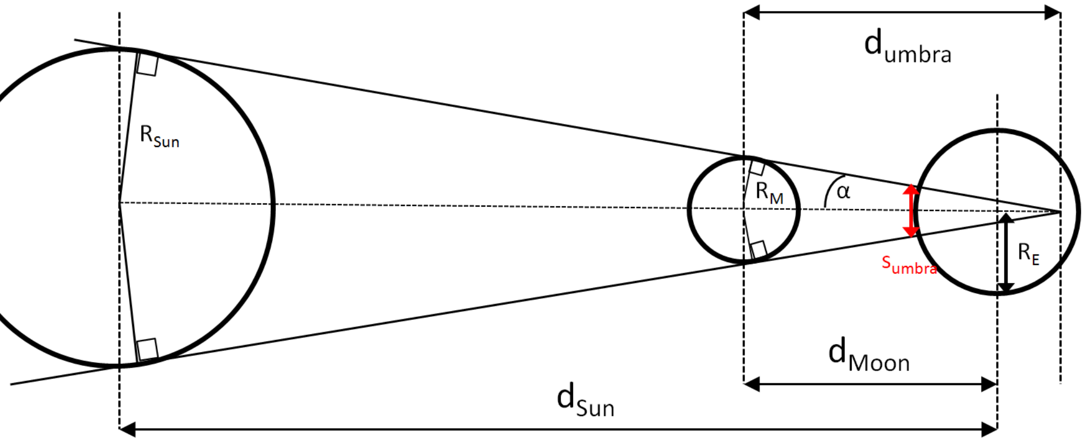
The value of H_0 derived from multi-messenger astronomy **is consistent** with the value derived by the HST using Cepheid variables [1] [1]

[Allow uncertainties added in quadrature, giving $\Delta H_0 = 9.2$ km s^{-1} Mpc^{-1}]

Q3 - Great American Eclipse [Total = 25]

a. *Calculate the width of the path of totality at GE (in km). You may use the approximation that the Moon is directly overhead so that the shadow is circular, and small enough that the curvature of the Earth can be neglected.*

Suitable diagram [1]



From the apparent angular radii we can calculate the distance to the Moon and the Sun:

Moon: $16'03.4'' = 4.67 \times 10^{-3}$ rad $\therefore d_{\text{Moon}} = \frac{R_{\text{Moon}}}{\theta} = \frac{1737}{4.67\times10^{-3}} = 371\ 900$ km [1]

Sun: $15'48.7'' = 4.60 \times 10^{-3}$ rad $\therefore d_{\text{Sun}} = \frac{R_{\text{Sun}}}{\theta} = \frac{695700}{4.60\times10^{-3}} = 151\ 258\ 000$ km [1]

Angle of shadow: $\alpha = \sin^{-1}\left(\frac{R_{\text{Sun}}-R_{\text{Moon}}}{d_{\text{Sun}}-d_{\text{Moon}}}\right) = 4.60 \times 10^{-3}$ rad $(= 0.2635°)$ [1]

Shadow cone length: $d_{\text{umbra}} = \frac{R_{\text{Moon}}}{\sin\alpha} = 377\ 670$ km [1]

Full width of shadow: $s_{\text{umbra}} = 2\left(d_{\text{umbra}} - (d_{\text{Moon}} - R_{\text{Earth}})\right)\tan\alpha = 111.7$ km [1] [6]

[Allow s_{umbra} = 55.9 km if it is clear that the student recognises that's the radius of the circular shadow, otherwise they lose one mark. Allow use of small angle approximations throughout or other slight variations in the geometry resulting in a width within a few km of the correct answer]

b. *Show that at this point the shadow is moving across the Earth's surface at approximately 0.68 km s^{-1}. You may use the simplifying approximation that for short time intervals it can be considered as travelling with a constant latitude, and that the apparent movement of the Sun due to the Earth's orbit can be neglected.*

At GE, the latitude is $\theta = 36°58' = 36.967°$

$$v_{\text{rot}} = \frac{2\pi R_E \cos\theta}{T} = \frac{2\pi\times6371\times\cos 36.967}{24\times60\times60} \quad [1]$$

$$= 0.370 \text{ km s}^{-1} \quad [1]$$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \therefore v_{\text{Moon}} = \sqrt{GM_E\left(\frac{2}{d_{\text{Moon}}} - \frac{1}{a}\right)}$$

$$= \sqrt{6.67\times10^{-11}\times5.97\times10^{24}\left(\frac{2}{371900\times10^3} - \frac{1}{384400\times10^3}\right)} \quad [1]$$

$$= 1.051 \text{ km s}^{-1} \quad [1]$$

Hence: $v_{\text{rel}} = v_{\text{Moon}} - v_{\text{rot}} = 1.051 - 0.370 = 0.681$ km s^{-1} [1] [5]

[Since this is a 'show that' question, ensure the student has shown evidence that they have calculated the correct relative velocity (rather than simply quoted it) for the final mark]

c. *Hence calculate the duration of the eclipse. Give your answer to the nearest 0.1 s.*

$$t_{\text{eclipse}} = \frac{s_{\text{umbra}}}{v_{\text{rel}}} = \frac{111.74}{0.681} = 164.0 \text{ s} \quad (= 2 \text{ mins } 44.0 \text{ secs})$$ [1] [1]

[Answer must be to the nearest 0.1 s for the mark]

Despite many of the simplifying assumptions made, this is within a few seconds of the measured value of 2 mins 41.6 secs or the official predicted value of 2 mins 40.3 secs

d. *The town of Carbondale, Illinois, is the closest big town to the point of GD, with co-ordinates 37°44'N latitude and 89°13'W longitude. Assuming the path of maximum totality can be treated as linear as it passes through the region around GD and GE:*

i. *Calculate the co-ordinates of the closest point ("CP") to Carbondale on the path of maximum totality.*

Converting the co-ordinates of GE and GD:

GE: $x_1 = 87.67°, y_1 = 36.97°$ GD: $x_2 = 89.12°, y_2 = 37.58°$

Equation of a straight line is $y = mx + c$, so:

$$m = \frac{x_2 - x_1}{y_2 - y_1} = 0.427$$ [1]

$$c = y_1 - mx_1 = -0.448$$ [1]

[Allow reversed signs of m and c so long as internally consistent]

Equation of the line passing through Cardondale (with co-ordinates x_C, y_C) and perpendicular to the initial line at CP (with co-ordinates x_{CP}, y_{CP}) must satisfy

$$mx_{CP} + c = -\frac{1}{m}(x_{CP} - x_C) + y_C$$ [1]

$$\therefore x_{CP} = \frac{x_C + my_C - mc}{m^2 + 1} = 89.26° \ (= 89°15.3')$$ [1]

$$y_{CP} = mx_{CP} + c = 37.64° \ (= 37°38.6')$$ [1] [5]

ii. *Calculate the distance (in km) between Carbondale and CP.*

We can calculate the angular distance, utilising Pythagoras' theorem since they are very close together:

$$\theta_{\text{dist}} = \sqrt{(x_{CP} - x_C)^2 + (y_{CP} - y_C)^2} = 0.0987° = 0.0017 \text{ rad}$$ [1]

Using the small angle approximation (and ignoring the surface curvature on this scale):

$$d = \theta_{\text{dist}} \times R_E = 0.0017 \times 6371 = 11.0 \text{ km}$$ [1] [2]

iii. *Hence calculate (to the nearest 0.1 s) how much shorter totality was for residents in Carbondale compared with CP. Take the duration at CP to be the same as GD, the width of the path to be 0.5 km less than for GE (so the Moon's shadow is elliptical), and the speed of the Moon's shadow to have only been affected by the change in latitude.*

At CP: $t_{\text{eclipse,CP}} = 164.1\text{ s}$ $\quad s_{\text{umbra,CP}} = s_{\text{umbra}} - 0.5 = 111.24\text{ km}$

$v_{\text{Moon}} = 1.051\text{ km s}^{-1}$ (same as before)

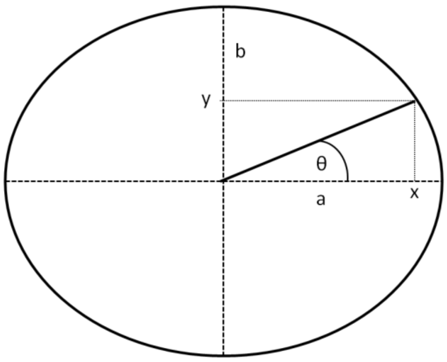
$v_{\text{rot,CP}} = \frac{2\pi R_E \cos(y_{CP})}{T} = 0.367\text{ km s}^{-1} \therefore v_{\text{rel,CP}} = 0.684\text{ km s}^{-1}$ [1]

Eclipse shadow is now elliptical with:

$\text{Minor axis} = s_{\text{umbra,CP}} = 111.24\text{ km}$

$\text{Major axis} = t_{\text{eclipse,CP}} \times v_{\text{rel,CP}} = 112.36\text{ km}$ [1]

For an ellipse, $x = a\cos\theta$ and $y = b\sin\theta$



Hence if $y = 11.0\text{ km}$ and $b = \frac{111.24}{2}$ $\Rightarrow \theta = 11.38° \therefore x = 55.07\text{ km}$ [2]

[First mark is for an appropriate diagram and use of the equation of an ellipse]

So the length of the shadow over Carbondale $= 2x = 110.1\text{ km}$ [1]

$\therefore t_{\text{eclipse,C}} = \frac{110.1}{0.684} = 160.9\text{ s}$

$\therefore \Delta t = 164.1 - 160.9 = 3.2\text{ s}$ [1] [6]

[Answer must be to the nearest 0.1 s for the mark]

Carbondale is lucky enough that it will also be on the path of totality of the next eclipse to cross the mainland of the USA on 8th April 2024!

Q4 - Super-Earths and Planet Nine [Total = 25]

a. *Based on the Marcy et al. (2014) model, Planet Nine is most likely to be a gas dwarf with a thick gaseous envelope. Calculate R_P (in units of R_E) for Planet Nine using this model.*

$$\frac{R_P}{R_E} = \left(\frac{M_P/M_E}{2.69}\right)^{1/0.93} = \left(\frac{10}{2.69}\right)^{1/0.93} \quad [1]$$
$$= 4.10\ R_E \quad [1] \quad [2]$$

[Allow 1 mark max if they forget to raise to the power of 1/0.93, giving $R_P = 3.72\ R_E$]

b. *Planetary formation models suggest such a gas dwarf would have a solid rocky core the size of the Earth, with similar composition and density. Calculate a simple estimate of the atmospheric pressure on the rocky surface. Compare your answer to the atmospheric pressure at sea level on the Earth. (p_E = 100 kPa)*

$$V_{\text{atm}} = V_{\text{planet}} - V_{\text{core}} = \frac{4}{3}\pi(4.10\ R_E)^3 - \frac{4}{3}\pi R_E^3$$
$$= \frac{4}{3}\pi(4.10^3 - 1^3)(6.37\times10^6)^3 = 7.37\times10^{22}\ \text{m} \quad [1]$$
$$M_{\text{atm}} = M_{\text{planet}} - M_{\text{core}} = 9\ M_E\ \ (= 5.37\times10^{25}\ \text{kg}) \quad [1]$$
$$\rho_{\text{atm}} = \frac{M_{\text{atm}}}{V_{\text{atm}}} = \frac{5.37\times10^{25}}{7.37\times10^{22}} = 729\ \text{kg m}^{-3}\ \ (= 0.73\ \text{g cm}^{-3}) \quad [1]$$
$$p_{\text{atm}} = \rho_{\text{atm}} \times g \times h_{\text{atm}} = 729 \times 9.81 \times (4.10 - 1)R_E$$
$$= 1.41\times10^{11}\ \text{Pa} \quad [1] \quad [4]$$

This is about a million times larger than atmospheric pressure on Earth [1] [1]

[Allow other models for atmospheric pressure, such as taking the weight of the whole atmosphere and applying it to the surface area of an Earth-sized core, giving 1.03×10^{12} Pa. Don't expect to see any attempt to model the variation in *g* or ρ with height since it only asked for a simple estimate]

c. *Although In practice the transition between rocky and gaseous planets is not likely to be sharply at 1.5 R_E, with some planets of both types existing above and below the limit. Calculate R_P (in units of R_E) for Planet Nine if it was a rocky super-Earth. How does its average density compare to an Earth-sized rocky exoplanet?*

$$\rho = 1000\left[2.32 + 3.18\left(\frac{R_P}{R_E}\right)\right] = \frac{M_P}{\frac{4}{3}\pi R_P^3}$$
$$\therefore \frac{3180}{R_E}R_P^4 + 2320R_P^3 - \frac{M_P}{\frac{4}{3}\pi} = 0 \quad [1]$$

Only positive real root is $R_P = 1.20\times10^7\ \text{m} = 1.88\ R_E$ [3] [4]

$$\rho_{\text{Earth}} = 2.32 + 3.18 = 5.50\ \text{g cm}^{-3} \quad [1]$$
$$\rho_{\text{av}} = 2.32 + (3.18\times1.88) = 8.30\ \text{g cm}^{-3} = 1.51\ \rho_{\text{Earth}} \quad [1] \quad [2]$$

[Allow any valid method to solve the quartic, such as using their graphical calculator to plot the graph and find roots, or even a trial and error iteration. If when constructing the quartic they forget to convert the densities into kg m^{-3} (leading to $R_P = 7.20\times10^7$ m = 11.3 R_E) then only lose 1 mark] *d. Verify that the above rocket design is sufficient to escape from Earth but **not** sufficient to escape from the surface of a rocky Planet Nine.*

$$v_{\max} = v_e \ln\frac{m_0}{m_1} = 4.46 \ln\left(\frac{1}{1-0.96}\right) = 14.4 \text{ km s}^{-1}$$ [1]

$$v_{\text{esc,Earth}} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2\times 6.67\times 10^{11}\times 5.97\times 10^{24}}{6.37\times 10^{6}}} = 11.2 \text{ km s}^{-1}$$ [1]

$$v_{\text{esc,PN}} = \sqrt{\frac{2GM_P}{R_P}} = \sqrt{\frac{2\times 6.67\times 10^{11}\times (10\times 5.97\times 10^{24})}{(1.88\times 6.37\times 10^{6})}} = 25.8 \text{ km s}^{-1}$$ [1] [3]

$$v_{\text{esc,Earth}} < v_{\max} < v_{\text{esc,PN}}$$ [1] [1]

So this rocket design is able to allow us to escape Earth's gravity, but is far short of what is needed for Planet Nine (which has implications for any probe that ever lands on it, should it exist)
[Allow the comparisons to be done in terms of gravitational potential rather than escape velocities. If the student **did not** get a value of R_P in part c. then allow them to use their value of R_P from part a. for full credit (giving $v_{esc,PN}$ = 17.5 km s^{-1} for R_P = 4.10 R_E)]

e. Calculate the maximum value of R_P (for a rocky exoplanet) above which any alien civilization would be unable to escape their planet's gravity using simple chemical rocket propulsion systems. (They could of course still have orbital satellites, since the speeds for planetary orbit are lower)

$$M_P = 1000\left[2.32 + 3.18\left(\frac{R_{P,\max}}{R_E}\right)\right] \times \frac{4}{3}\pi R_{P,\max}^3$$

$$v_{\max}^2 = \frac{2GM_P}{R_{P,\max}} = \frac{2G}{R_{P,\max}} \times 1000\left[2.32 + 3.18\left(\frac{R_{P,\max}}{R_E}\right)\right] \times \frac{4}{3}\pi R_{P,\max}^3$$ [1]

$$\therefore \frac{3180}{R_E} R_{P,\max}^3 + 2320 R_{P,\max}^2 - \frac{v_{\max}^2}{\frac{8}{3}\pi G} = 0$$ [1]

Only positive real root is $R_{P,\max} = 7.73 \times 10^6$ m $\quad (= 1.21\ R_E)$ [3] [5]

[Allow any valid method to solve the cubic, such as using their graphical calculator to plot the graph and find roots, or even a trial and error iteration. If when constructing the cubic they forget to convert the densities into kg m^{-3} (leading to $R_{P,max}$ = 8.89 × 10^7 m = 14.0 R_E) then only lose 1 mark. Forgetting to express v_{max} in m s^{-1} leads to nonsensical values of $R_{P,max}$ (less than R_E) given they should only be looking at the positive roots]

f. In the Marcy et al. (2014) model, above R_P = 1.5 R_E the density of gas dwarfs rapidly decreases with radius. By looking at the piecewise function explain why this does not improve the situation for any alien civilization hoping to explore their solar system (ignoring that such planets are far less likely to be habitable). You do not need to calculate any new escape velocities.

In the model for gas dwarfs $M_P \propto R_P^{0.93} \approx M_P \propto R_P$ [1]

Since $v_{\text{esc}} \propto \sqrt{\frac{M_P}{R_P}}$ then if $M_P \propto R_P \Rightarrow v_{\text{esc}} \sim \text{constant}$ [1]

(So the escape velocity will be roughly the same for all gas dwarfs, and approximately the same as the escape velocity for a rocky planet with R_P = 1.5 R_E)

Rocket design gives insufficient thrust at R_P > 1.21 R_E, so will be insufficient at 1.5 R_E, and hence will be insufficient for all gas dwarfs too [1] [3]

END OF PAPER