Round 2 Solutions - January 2017

These solutions written about the problems and do not constitute what the student might be expected to write as a solution in an exam.

Some changes occur during marking and those changes may not be included here.

- Solutions for Question 1 are not provided

Correction

In Q4(b)(ii) there is a spurious μ in the solution suggested to the differential equation derived. This is a hangover from the original setting of the question in terms of reduced mass, so should have been replaced by m_p .

The formula they find in (b)(iii) for Kepler's 3rd law may end up having a μ in it, and students will at least have come across the 3rd law in some detail and most of them will have seen it in terms of the semi-major axis in the BAAO paper - meaning that there should not be too much confusion over its use in parts (c) and (d).

However, be sure to bear this error in mind when looking at their attempts at Q4.

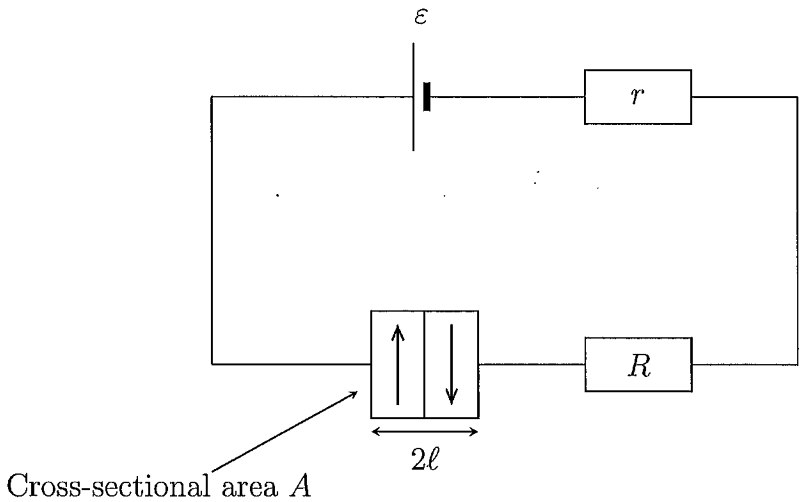
This is a set of solutions and explanations rather than a markscheme. Students will do well on one or two questions and very badly or not even attempt others. Give marks for progress. Be consistent but follow your judgement. We are selecting out the top students from this, not trying to obtain an exact ranking for everyone. We have awards in bands because with these questions it is not possible to have a precise mark scheme.
If in doubt email Robin Hughes on RH584@cam.ac.uk

British Physics Olympiad

Round 2 Solutions Qs 2-5

Qu2. Magnetoresistance

(a) The circuit is as follows:



✓ (1)

(b) Consider a spin-down electron as it moves through the first and then the second layer (from right to left). It will experience a layer of magnetisation parallel to its own followed by one anti-parallel. The current due to these electrons will therefore experience a resistance of

$$\begin{aligned} R_{\text{tot}} &= R_{\text{p}} + R_{\text{a}} \\ &= \frac{\ell}{A}(\rho_{\text{p}} + \rho_{\text{a}}) \end{aligned}$$

✓ ✓

Spin-up electrons will experience the opposite (anti-parallel followed by parallel), giving the same resistance experienced overall. Since the currents can be regarded as being independent, the resistances experienced add in parallel, giving:

$$\begin{aligned} R_{\text{combined}} &= \frac{R_{\text{tot}}}{2} \\ &= \frac{\ell}{2A}(\rho_{\text{p}} + \rho_{\text{a}}) \end{aligned}$$

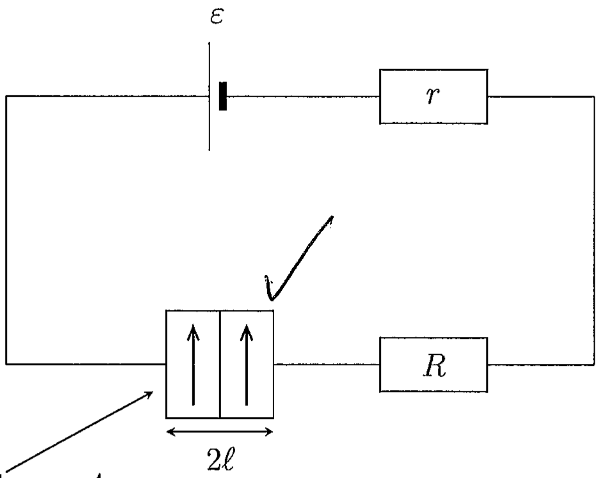
✓

The total circuit resistance is therefore

$$\begin{aligned} R_{\text{tot}} &= R_{\text{combined}} + R + r \\ &= \frac{\ell}{2A}(\rho_{\text{p}} + \rho_{\text{a}}) + R + r \end{aligned}$$

✓ (4)

(c) The circuit is now as follows:



Considering either a spin-up or spin-down electron as it moves through these two layers, it experiences the same magnetisation throughout (either parallel or anti-parallel), for a total length of 2ℓ. Electrons with their spin parallel to the magnetic field will therefore experience a resistance of

$$R_\mathrm{p} = \frac{\rho_\mathrm{p} 2\ell}{A}$$

and likewise for electrons with their spin anti-parallel. Since the currents can be regarded as being independent, the resistances experienced add in parallel, giving:

$$\begin{aligned} \frac{1}{R_\mathrm{combined}} &= \frac{1}{R_\mathrm{p}} + \frac{1}{R_\mathrm{a}} \\ &= \frac{A}{2\rho_\mathrm{p}\ell} + \frac{A}{2\rho_\mathrm{a}\ell} \\ &= \frac{A}{2\ell}\frac{\rho_\mathrm{p}+\rho_\mathrm{a}}{\rho_\mathrm{p}\rho_\mathrm{a}} \end{aligned}$$

and hence a circuit resistance of

$$\begin{aligned} R_\mathrm{tot} &= R_\mathrm{combined} + R + r \\ &= \frac{2\ell}{A}\frac{\rho_\mathrm{p}\rho_\mathrm{a}}{\rho_\mathrm{p}+\rho_\mathrm{a}} + R + r \end{aligned}$$

4

(d) **(i)** With $\rho_\mathrm{p} \ll \rho_\mathrm{a}$ we get

$$R_\mathrm{(b)} \approx \frac{\ell}{2A}\rho_\mathrm{a} + R + r$$

and

$$R_\mathrm{(c)} \approx \frac{2\ell}{A}\rho_\mathrm{p} + R + r$$

so

$$
\begin{aligned}
\Delta I &= I_{(\mathrm{c})} - I_{(\mathrm{b})} \\
&= \frac{\varepsilon}{R_{(\mathrm{c})}} - \frac{\varepsilon}{R_{(\mathrm{b})}} \\
&= \varepsilon\left(\frac{1}{\frac{2\ell}{A}\rho_{\mathrm{p}} + R + r} - \frac{1}{\frac{\ell}{2A}\rho_{\mathrm{a}} + R + r}\right)
\end{aligned}
$$

3

(ii) For fixed ε, r, R, ρ_{p} and ρ_{a} ($\rho_{\mathrm{p}} \ll \rho_{\mathrm{a}}$), the change in current depends on the physical dimensions only in the combination $x = \ell/A$ as:

$$
\Delta I = \varepsilon\left(\frac{1}{2x\rho_{\mathrm{p}} + R'} - \frac{1}{\frac{1}{2}x\rho_{\mathrm{a}} + R'}\right)
$$

0 → 3 depending on progress..

where $R' = R + r$. This is extremized when

$$
\begin{aligned}
\frac{\mathrm{d}\Delta I}{\mathrm{d}x} &= 0 \\
\Rightarrow -\varepsilon\left(\frac{2\rho_{\mathrm{p}}}{(2x\rho_{\mathrm{p}} + R')^2} - \frac{\frac{1}{2}\rho_{\mathrm{a}}}{(\frac{1}{2}x\rho_{\mathrm{a}} + R')^2}\right) &= 0 \\
\Rightarrow 2\rho_{\mathrm{p}}(x\rho_{\mathrm{a}}/2 + R')^2 - \frac{1}{2}\rho_{\mathrm{a}}(2x\rho_{\mathrm{p}} + R')^2 &= 0 \\
\Rightarrow \frac{\rho_{\mathrm{p}}\rho_{\mathrm{a}}^2}{2}\left(x + \frac{2R'}{\rho_{\mathrm{a}}}\right)^2 - 2\rho_{\mathrm{a}}\rho_{\mathrm{p}}^2\left(x + \frac{R'}{2\rho_{\mathrm{p}}}\right)^2 &= 0 \\
\Rightarrow \rho_{\mathrm{a}}\left(x^2 + 4\frac{R'}{\rho_{\mathrm{a}}}x + 4\left(\frac{R'}{\rho_{\mathrm{a}}}\right)^2\right) - 4\rho_{\mathrm{p}}\left(x^2 + \frac{R'}{\rho_{\mathrm{p}}}x + \left(\frac{R'}{4\rho_{\mathrm{p}}}\right)^2\right) &= 0 \\
\Rightarrow x^2(\rho_{\mathrm{a}} - 4\rho_{\mathrm{p}}) + 4xR' - 4xR' + 4\frac{R'^2}{\rho_{\mathrm{a}}} - \frac{R'^2}{\rho_{\mathrm{p}}} &= 0 \\
\Rightarrow x^2(\rho_{\mathrm{a}} - 4\rho_{\mathrm{p}}) - \frac{R'^2}{\rho_{\mathrm{p}}\rho_{\mathrm{a}}}(\rho_{\mathrm{a}} - 4\rho_{\mathrm{p}}) &= 0
\end{aligned}
$$

That is (assuming $\rho_{\mathrm{a}} \gg \rho_{\mathrm{p}}$ means that certainly $\rho_{\mathrm{a}} > 4\rho_{\mathrm{p}}$ – *i.e.* $\rho_{\mathrm{a}} \neq 4\rho_{\mathrm{p}}$)

$$
\boxed{x = \frac{R'}{\sqrt{\rho_{\mathrm{p}}\rho_{\mathrm{a}}}} = \frac{R + r}{\sqrt{\rho_{\mathrm{p}}\rho_{\mathrm{a}}}}}
$$

Differentiating again:

$$
\frac{\mathrm{d}^2\Delta I}{\mathrm{d}x^2} = \varepsilon\left(\frac{8\rho_{\mathrm{p}}^2}{(2x\rho_{\mathrm{p}} + R')^3} - \frac{\frac{1}{2}\rho_{\mathrm{a}}^2}{(\frac{1}{2}x\rho_{\mathrm{a}} + R')^3}\right)
$$

so that

$$\left.\frac{d^2\Delta I}{dx^2}\right|_{x=\frac{R'}{\sqrt{\rho_p\rho_a}}} = \varepsilon\left(\frac{8\rho_p^2}{(2x\rho_p+R')^3} - \frac{\frac{1}{2}\rho_a^2}{(\frac{1}{2}x\rho_a+R')^3}\right)$$

$$= \frac{\varepsilon}{R'^3}\left(\frac{8\rho_p^2}{\left(2\sqrt{\frac{\rho_p}{\rho_a}}+1\right)^3} - \frac{4\rho_a^2}{\left(\sqrt{\frac{\rho_a}{\rho_p}}+2\right)^3}\right)$$

$$= \frac{\varepsilon\rho_p}{R'^3}\left(\frac{8\rho_p}{\left(1+2\sqrt{\frac{\rho_p}{\rho_a}}\right)^3} - \sqrt{\frac{\rho_p}{\rho_a}}\frac{4\rho_a}{\left(1+2\sqrt{\frac{\rho_p}{\rho_a}}\right)^3}\right)$$

$$= -\frac{8\varepsilon\rho_p^2}{R'^3}\frac{\sqrt{\frac{\rho_a}{\rho_p}}}{\left(1+2\sqrt{\frac{\rho_p}{\rho_a}}\right)^3}\left(\frac{1}{2}-\sqrt{\frac{\rho_p}{\rho_a}}\right)$$

which is manifestly less than zero since $\rho_p \ll \rho_a$ (again assuming $\rho_a \gg \rho_p$ means that certainly $\rho_a > 4\rho_p$), showing that this value of x gives a maximum change in current.

(e) When **writing** to disk, the read/write head would generate a (strong) magnetic field, creating a magnetic field pattern in the disk surface as it is rotated underneath to store information. When **reading**, the magnetised pattern on the disk read surface would rotate past the read/write head, generating (inducing) a current in the coil. When reading magnetic field patterns, we would therefore want a maximum change in current generated between areas of opposite magnetisation so that different states, and hence different 'bits' of information can be detected. We would therefore want the physical dimensions of the head to be optimised such that $\frac{\ell}{A} = \frac{R+r}{\sqrt{\rho_p\rho_a}}$ as in part **(d)** above. However, coupled with this we would also want to minimise the overall 'width' of the head so that it reads only the desired magnetisation area at one time, and not neighbouring ones at the same time. As a certain magnetic field strength would be required to write to the disk it would also be important to reduce the distance between the read/write head and the disk. ✓ ✓ ✓ ✓

A mark for a relevant comment, besides these ones listed.

5

Qu3. Dual Radioactive Decay

(a) The number of ion pairs is

$$n_{\text{pairs}} = \frac{6.5 \times 10^6}{15.6} = 4.17 \times 10^5$$

Assuming that a single electron is liberated in each ionization event, the number of ion pairs per second is

$$\frac{17 \times 10^{-9}}{1.6 \times 10^{-19}} = 1.06 \times 10^{10}\,\text{s}^{-1}$$

The number of alpha particles per second is therefore

$$\frac{1.06 \times 10^{10}}{4.17 \times 10^5} = \underline{2.55 \times 10^5\,\text{s}^{-1}}$$

(for close values)

3

(b) As radioactive decay is a random process, if the number of decays per second (activity) is A, the variation in this will be of the order of $\sqrt{A}$. If I is the current, then from **(a)**

$$I = An_{\text{pairs}}e$$

and

$$\delta I = \delta A n_{\text{pairs}} e = \sqrt{A} n_{\text{pairs}} e$$

that is

$$\begin{aligned} \frac{\delta I}{I} &= \frac{\sqrt{A}}{A} \\ \Rightarrow \delta I &= \frac{\sqrt{A}}{A} I \\ &= \frac{\sqrt{2.55 \times 10^5}}{2.55 \times 10^5} \times 17 \times 10^{-9} \\ &\approx 0.2\% \times 17 \times 10^{-9} \\ &= \underline{0.034 \times 10^{-9}\,\text{A}} \end{aligned}$$

3

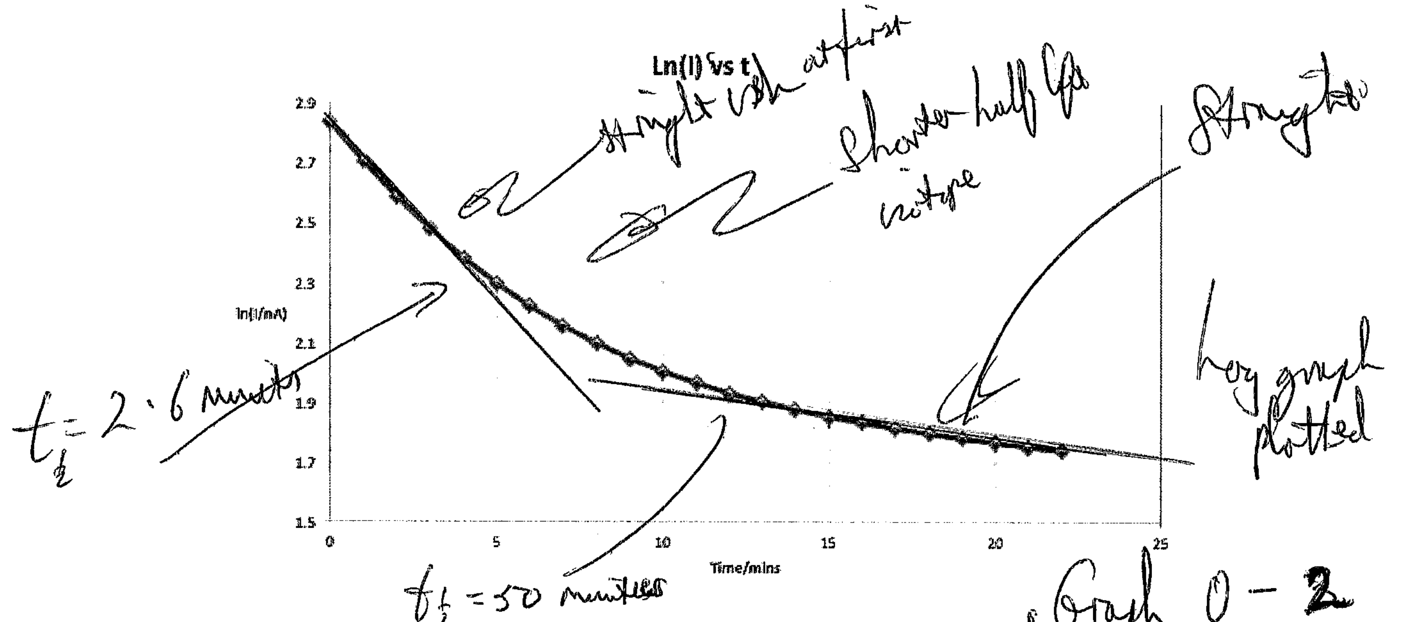
This is approximately three orders of magnitude smaller than the current itself, so will not be detected straightforwardly.

(c) A precision measurement of the half-life of a radioactive isotope by conventional techniques requires the observations to be extended over a period of time comparable to the half-life. Even for moderately long half-lives this is not only time consuming but introduces many experimental difficulties: The sensitivity of the detector may change in time, the source detector geometry may not be reproducible, and the background may vary during the experiment. More important, a slight variation in apparent half-life due to impurities in the sample is difficult to detect during the course of the observations. The Balanced Ion Chamber technique does not suffer from the above difficulties. By using large cylindrical ion chambers and by introducing the source along the symmetry axis, the counters are made less sensitive to changes in geometry than conventional end window counters with external sources. Any change in background will produce an equal effect in both chambers. Hence, the current reading which is the difference current remains unchanged. A half-life in the region of 1 year can be measured to an accuracy of 1% by observations extending over a period of roughly 10 days. This relatively rapid measurement permits several independent half-life determinations to be made on a given sample, and any slight variation with time of the measured half-life is readily detectable.

Max 4 for any good statements.

4

(d) Here is a graph of $\ln(I)$ against time (with I in nano-amps). See table in part **(f)** for values, though if the currents given in the question are converted into amperes before logarithms are taken, the logarithmic values will be offset by $\ln(1 \times 10^{-9}/\mathrm{A}) = -\ln(1 \times 10^{9}/\mathrm{A}) \approx -20.723$. The final results for values of λ and half-lives should of course be unchanged. From **(b)** we



Graph 0 - 2

have $I = An_{\text{pairs}}e$. Now, at a given point in time, with two isotopes present (S and L, say), there will be a total number of molecules

Features 0 - 3

Max 5

$$\begin{aligned} N(t) &= N_{\mathrm{S}} + N_{\mathrm{L}} \\ &= N_{\mathrm{S}0}e^{-\lambda_{\mathrm{S}}t} + N_{\mathrm{L}0}e^{-\lambda_{\mathrm{L}}t} \end{aligned}$$

assuming that each isotope decays independently. So

$$\begin{aligned} A &= A_{\mathrm{S}} + A_{\mathrm{L}} \\ &= \lambda_{\mathrm{S}}N_{\mathrm{S}} + \lambda_{\mathrm{L}}N_{\mathrm{L}} \\ &= \lambda_{\mathrm{S}}N_{\mathrm{S}0}e^{-\lambda_{\mathrm{S}}t} + \lambda_{\mathrm{L}}N_{\mathrm{L}0}e^{-\lambda_{\mathrm{L}}t} \end{aligned}$$

Unlikely.

With $n_{\text{pairs}}e = c$ this then implies

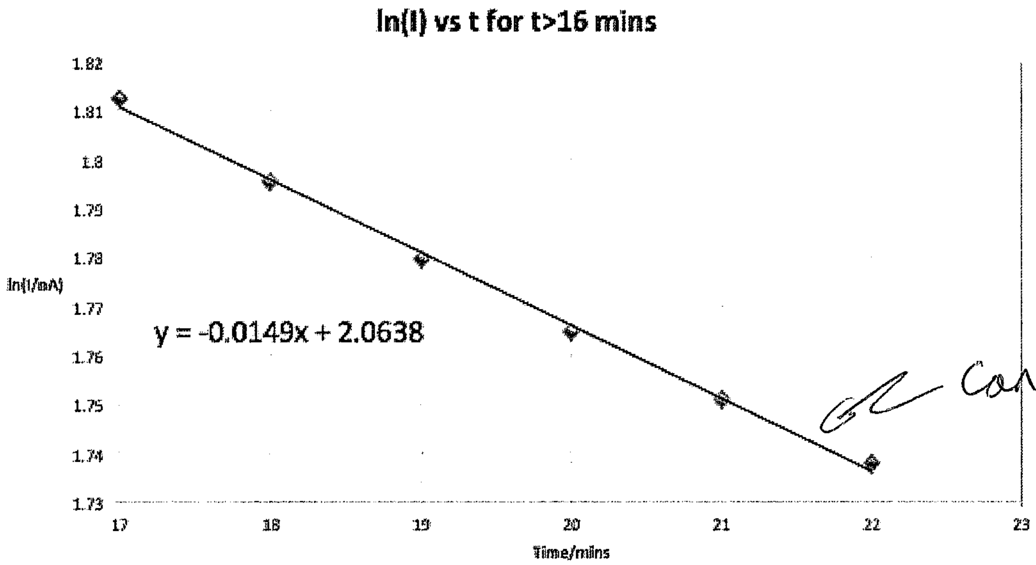
$$\begin{aligned} I &= cA \\ &= c(\lambda_{\mathrm{S}}N_{\mathrm{S}0}e^{-\lambda_{\mathrm{S}}t} + \lambda_{\mathrm{L}}N_{\mathrm{L}0}e^{-\lambda_{\mathrm{L}}t}) \end{aligned}$$

Now without loss of generality let $\lambda_{\mathrm{S}} > \lambda_{\mathrm{L}}$ so that S has the shorter half-life. Initially, I (and hence $\ln(I)$) decreases in a nonlinear fashion due to the exponential decay of the activities of both isotopes. However, after a certain amount of time, $\lambda_{\mathrm{S}}t$ will be large enough so that $e^{-\lambda_{\mathrm{S}}t}$ will become small such that the term $c\lambda_{\mathrm{S}}N_{\mathrm{S}0}e^{-\lambda_{\mathrm{S}}t}$ is virtually undetectable – *i.e.* $e^{-\lambda_{\mathrm{S}}t} \approx 0$. In this case we will have

$$\begin{aligned} \ln(I) &= \ln(c) + \ln(\lambda_{\mathrm{S}}N_{\mathrm{S}0}e^{-\lambda_{\mathrm{S}}t} + \lambda_{\mathrm{L}}N_{\mathrm{L}0}e^{-\lambda_{\mathrm{L}}t}) \\ &\approx \ln(c\lambda_{\mathrm{L}}N_{\mathrm{L}0}) - \lambda_{\mathrm{L}}t \end{aligned}$$

Therefore a graph of $\ln(I)$ vs. time should decrease nonlinearly to begin with, but after a certain amount of time (approximately 15 minutes by the look of the graph) it should resemble a straight line graph with a negative gradient; the isotope with the shorter half-life is no-longer significantly contributing to the current.

(e) The values for 17 minutes and greater (see table in part (f)) are plotted below. Since the gradient



Can use the first graph.

must be equal to $-\lambda_{\rm L}$, this gives $\lambda_{\rm L} = 0.0149$ and hence a half life of

$$\begin{aligned} T_{\rm L\frac{1}{2}} &= \frac{\ln(2)}{\lambda_{\rm L}} \\ &\approx \underline{46.5\,\text{min}} \end{aligned}$$

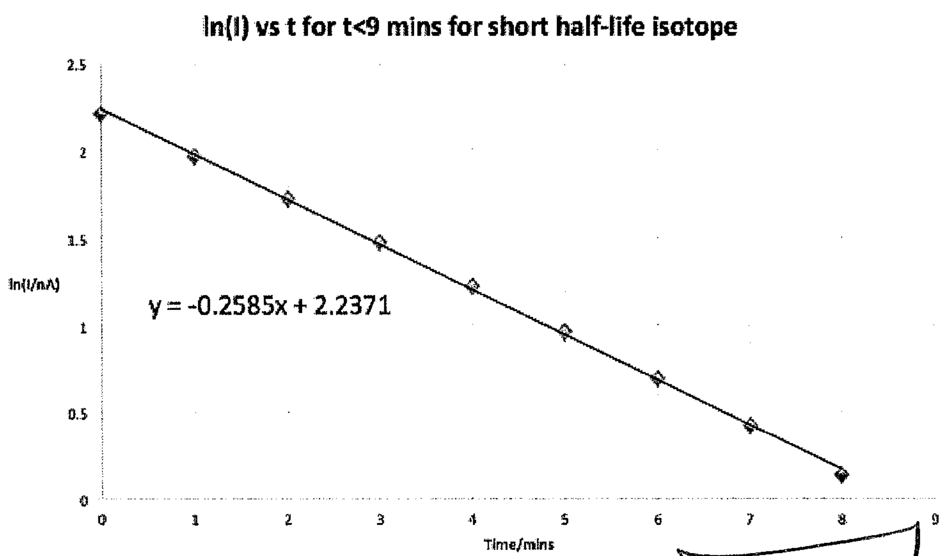
± 5 minutes

1

(f) The formula for the long half-life component of the gas $\ln(I_L) = -0.0149t + 2.0638$ (from the graph for $t > 17\,\text{min}$) then gives values of $I_L = e^{-0.0149t+2.0638}$ and $I_S = I_{\text{total}} - I_L$ *etc.*

Time /min	$\ln(I_L/\text{nA})$	$I_L/\text{A} \times 10^{-9}$	$I_S/\text{A} \times 10^{-9}$	$\ln(I_S/\text{nA})$
0	2.833	7.876	9.124	2.211
1	2.703	7.760	7.170	1.970
2	2.584	7.645	5.616	1.726
3	2.477	7.532	4.380	1.477
4	2.381	7.421	4.400	1.224
5	2.296	7.311	2.623	0.964
6	2.220	7.203	2.009	0.698
7	2.154	7.096	1.525	0.422
8	2.096	6.991	1.146	0.136
9	2.046	6.888	0.849	-0.164
10	2.002	6.786	0.618	-0.481
11	1.964	6.686	0.441	-0.819
12	1.930	6.587	0.305	-1.187
13	1.901	6.489	0.203	-1.594
14	1.875	6.393	0.128	-2.058
15	1.852	6.299	0.073	-2.611
16	1.831	6.206	0.036	-3.323
17	1.813	6.114	0.012	-4.428
18	1.795	6.023	-0.002	N/A
19	1.780	5.934	-0.007	N/A
20	1.765	5.847	-0.006	N/A
21	1.751	5.760	0.0002	-8.394
22	1.738	5.675	0.010	-4.577

For short times (say up to 8 mins) $\ln(I_S)$ can be plotted as below. Since it is expected that



$\boxed{2}$ marks for result and the analyses. [Conceive the first graph to obtain the results.

$I_S = c\lambda_S N_{S0} e^{-\lambda_S t}$, the gradient of an $\ln(I_S)$ against time graph should be equal to $-\lambda_S$. The graph above therefore gives

$$\begin{aligned} \lambda_S &= 0.259 \\ \Rightarrow T_{S\frac{1}{2}} &= \frac{\ln(2)}{\lambda_S} \\ &\approx \underline{2.7\,\text{min}} \end{aligned}$$

($2\frac{1}{2}$ -3 minutes)

(g) No nuclei of type X present initially, so in both cases curve has same y-intercept as in part **(d)** (approx 2.8). Then

(i) If $\lambda_X > \lambda_S$, $T_{X\frac{1}{2}} < T_{S\frac{1}{2}}$. Initially, decay of S produces X which then decays rapidly giving an initial rise to a peak followed by a smooth decay to long-time behaviour as in **(d)**. Time-frames for each component to effectively decouple can be roughly estimated by (say) the time taken for the current to fall to a hundredth of its initial value. Since for the decay of any isotope

$$t = -\frac{1}{\lambda}\ln\left(\frac{N}{N_0}\right)$$

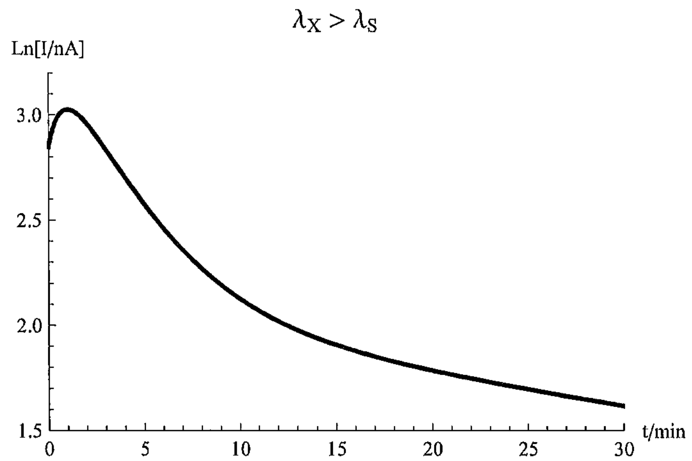
we may estimate the time after which S effectively ceases to affect the decay as

$$t_S = -\frac{1}{0.2585}\ln\left(\frac{1}{100}\right) \approx 18\,\text{mins}$$

This makes sense as the graph plotted in **(d)** has a linear section beginning at about 15 minutes (we used 17 minutes for the graph in **(e)**). Taking λ_X as being roughly 5 times λ_S ($\lambda_S = 1.2$ here for the sake of concreteness):

$$t_X = -\frac{1}{1.2}\ln\left(\frac{1}{100}\right) \approx 3\,\text{mins}$$

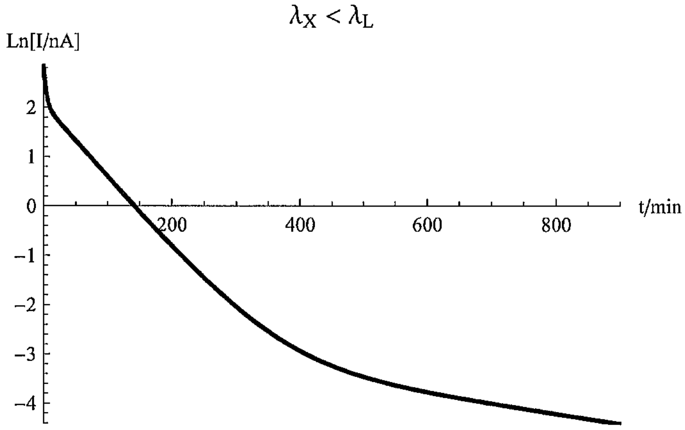
So, taking this all together, y-intercept is approximately 2.8, then there is an initial peak followed by a nonlinear decay where S and L both contribute before the graph becomes linear after about 15-20 mins.



✓ (1)

(ii) If $\lambda_X < \lambda_S$, $T_{X\frac{1}{2}} > T_{S\frac{1}{2}}$. In this case X dominates the long-time behaviour. The first two sections of the graph are therefore roughly as in **(d)**, with a final linear section taking over after approximately

$$t_L = -\frac{1}{0.0149}\ln\left(\frac{1}{100}\right) \approx 300\,\text{mins}$$



✓ (1)

20

Extra information not expected in a student answer:
To derive the relationship that describes these curves, consider first the differential equation governing the population of isotope X. The number of nuclei (and hence activity/rate of decay) of this isotope decreases due to its decay, but increases due to the decay of isotope S:

$$\frac{dN_X}{dt} = -\lambda_X N_X + \lambda_S N_S$$

but of course $N_S = N_{S0}e^{-\lambda_S t}$, so

$$\frac{dN_X}{dt} = -\lambda_X N_X + \lambda_S N_{S0} e^{-\lambda_S t}$$

However, this is of the form

$$\frac{dN_X}{dt} + P(t)N_X = Q(t)$$

where $P(t) = \lambda_X$ (= const.) and $Q(t) = \lambda_S N_{S0} e^{-\lambda_S t}$. This type of differential equation can be solved by first multiplying through by the integrating factor

$$e^{\int P(t)dt} = e^{\int \lambda_X dt} = e^{\lambda_X t}$$

Multiplying the differential equation through by this factor gives

$$\begin{aligned}
\frac{dN_X}{dt}e^{\lambda_X t} + \lambda_X e^{\lambda_X t} N_X &= \lambda_S N_{S0} e^{-\lambda_S t} e^{\lambda_X t} \\
\Rightarrow \frac{d}{dt}\left(N_X e^{\lambda_X t}\right) &= \lambda_S N_{S0} e^{(\lambda_X - \lambda_S)t} \\
\Rightarrow N_X e^{\lambda_X t} &= \int \lambda_S N_{S0} e^{(\lambda_X - \lambda_S)t} dt \\
\Rightarrow N_X e^{\lambda_X t} &= \frac{\lambda_S N_{S0}}{\lambda_X - \lambda_S} e^{(\lambda_X - \lambda_S)t} + \text{const.}
\end{aligned}$$

Now at $t = 0$, $N_X = 0$ so the constant of integration is $-\frac{\lambda_S N_{S0}}{\lambda_X - \lambda_S}$. Rearranging gives

$$N_X = \frac{\lambda_S N_{S0}}{\lambda_X - \lambda_S}(e^{-\lambda_S t} - e^{-\lambda_X t})$$

Overall, therefore,

$$\begin{aligned} N(t) &= N_S + N_L + N_X \\ &= N_{S0}e^{-\lambda_S t} + N_{L0}e^{-\lambda_L t} + \frac{\lambda_S N_{S0}}{\lambda_X - \lambda_S}(e^{-\lambda_S t} - e^{-\lambda_X t}) \end{aligned}$$

and

$$\begin{aligned} A &= A_S + A_L + A_X \\ &= \lambda_S N_S + \lambda_L N_L + \lambda_X N_X \\ &= \lambda_S N_{S0}e^{-\lambda_S t} + \lambda_L N_{L0}e^{-\lambda_L t} + \frac{\lambda_S \lambda_X N_{S0}}{\lambda_X - \lambda_S}(e^{-\lambda_S t} - e^{-\lambda_X t}) \end{aligned}$$

Finally, since the alpha particles emitted all have the same energy, $I = cA$ again. Note that the y-intercepts of the graphs in **(e)** and **(f)** will allow cN_{S0} and cN_{L0} to be found (and hence N_{S0} and N_{L0}) via

$$\begin{aligned} \ln(cN_{S0}) &= 2.2371 \\ \Rightarrow cN_{S0} &= \frac{1}{\lambda_B}e^{2.2371} \\ &= \frac{1}{0.2585}e^{2.2371} \\ &= 36.233 \end{aligned}$$

and

$$\begin{aligned} \ln(cN_{L0}) &= 2.0638 \\ \Rightarrow cN_{L0} &= \frac{1}{\lambda_B}e^{2.0638} \\ &= \frac{1}{0.0149}e^{2.0638} \\ &= 528.58 \end{aligned}$$

An immensely long solution. Look for key stages in the students answers and award marks for a point made or equation achieved.

Qu4. Elliptical Orbits

(a) Re-write polar formula as $r + \varepsilon r\cos\theta = \alpha$ and, noting that $x = r\cos\theta$, re-cast as

$$\begin{aligned} r + \varepsilon x &= \alpha \\ \Rightarrow (\alpha - \varepsilon x)^2 &= r^2 \\ \Rightarrow (\alpha - \varepsilon x)^2 &= x^2 + y^2 \\ \Rightarrow x^2(1-\varepsilon^2) + 2\alpha\varepsilon x + y^2 &= \alpha^2 \end{aligned} \qquad (1)$$

Now complete the square in 'x', recalling that $f(x) = ax^2 + bx = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$, so that

$$\begin{aligned} (1-\varepsilon^2)\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2 - \frac{\alpha^2\varepsilon^2}{(1-\varepsilon^2)} + y^2 &= \alpha^2 \\ \Rightarrow (1-\varepsilon^2)\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2 + y^2 &= \frac{\alpha^2}{1-\varepsilon^2} \\ \Rightarrow \frac{\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2}{\frac{\alpha^2}{(1-\varepsilon^2)^2}} + \frac{y^2}{\frac{\alpha^2}{1-\varepsilon^2}} &= 1 \end{aligned} \qquad (2)$$

This is of the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

so the ellipse has

$$\boxed{\text{centre } (x_0, y_0) = \left(-\frac{\alpha\varepsilon}{1-\varepsilon^2}, 0\right)}$$

and

$$\boxed{\text{semi-major axis } a = \frac{\alpha}{1-\varepsilon^2}}$$

and

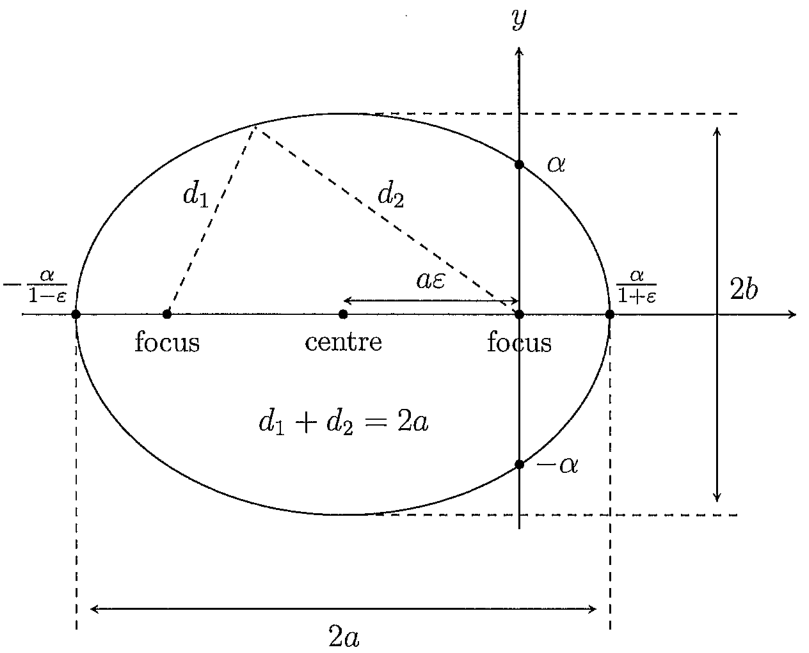
$$\boxed{\text{semi-minor axis } b = \frac{\alpha}{\sqrt{1-\varepsilon^2}}}$$

with

$$\begin{aligned} \text{eccentricity} &= \sqrt{1 - \frac{b^2}{a^2}} \\ &= \sqrt{1 - (1-\varepsilon^2)} \\ &= \varepsilon \end{aligned}$$

"Show that"
key points earn a mark
$\boxed{\text{Max } 4}$

Note that if $\varepsilon = 0$ then (2) reduces to $x^2 + y^2 = \alpha^2$, the equation of a circle (a sort of degenerate ellipse), while if $\varepsilon = 1$, (1) reduces to $y^2 = \alpha^2 - 2\alpha x$, the equation of a parabola. Therefore we must have $0 \le \varepsilon < 1$.



Annotated ellipse
Max 3.

(b) Note that as angular momentum is '$L = mr^2\dot{\theta}$', the angular momentum of the planet (assuming $M_S \gg m_p$ so that issues with reduced mass and centre of mass can be ignored) is $L = m_p r^2 \dot{\theta}$ (where $\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$).

(i) Newton's law of gravitation states the attractive force of gravity between two masses acts along a line joining their centres and is given in magnitude by

$$F = \frac{GM_S m_p}{r^2}$$

As the line of action of the gravitational force passes through the axis of rotation it produces no torque, and, assuming no other forces present, there is therefore **no net torque**. For rotational motion, Newton's second law states that the net torque is equal to the rate of change of angular momentum:

$$\tau_{\text{net}} = \frac{\mathrm{d}L}{\mathrm{d}t}$$

where L is the angular momentum of the planet. Since $\tau_{\text{net}} = 0$, $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$ and so $L = \text{constant}$.

Now the area element for an ellipse is in principle (this is easily derived by looking at areas in polar coordinates with a changing radius)

$$\delta A = \frac{1}{2}r^2\delta\theta + \frac{1}{2}r\delta r\delta\theta$$

but the second term is second order in small quantities, so can be neglected as $\delta\theta \to 0$. Thus, in this limit, we have:

$$\begin{aligned} \mathrm{d}A &= \frac{1}{2}r^2\mathrm{d}\theta \\ \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \end{aligned}$$

But the angular momentum of the planet is $L = m_p r^2 \dot{\theta}$ so

Key points. Max 4

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{L}{2m_p} = \text{constant}}$$

and hence $\delta A = \text{const.} \times \delta t$, *i.e.* equal areas are swept out in equal time intervals. **This is Kepler's second law.**

(ii) The energy of the system, given by

$$\begin{aligned} E &= \mathrm{KE} + \mathrm{PE} \\ &= \frac{1}{2} m_p v_{\text{radial}}^2 + \frac{1}{2} m_p v_{\text{tangential}}^2 - \frac{G M_S m_p}{r} \\ &= \frac{1}{2} m_p \dot{r}^2 + \frac{1}{2} m_p r^2 \dot{\theta}^2 - \frac{G M_S m_p}{r} \\ &= \frac{1}{2} m_p \dot{r}^2 + \frac{L^2}{2 m_p r^2} - \frac{k}{r} \end{aligned}$$

Max 3

where r is the distance of the planet from the Sun, and $k = G M_S m_p$. Note that since both the radius and angle change in an elliptical orbit, there are two contributions to the kinetic energy from motion in the radial direction and angular motion.

Extra information to connect with Kepler's first law:
In the following, $\mu = \frac{M_S m_p}{M_S + m_p}$ is the reduced mass of the system and can be set equal to m_p for the current case under consideration where $M_S \gg m_p$. Now we don't really want r as a function of time, but rather as a function of θ, so exchange t for θ via

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Then we have

$$\dot{r}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \dot{\theta}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \frac{L^2}{\mu^2 r^4}$$

so

$$\begin{aligned} E &= \frac{L^2}{2\mu r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} \\ \Rightarrow \left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) &= \frac{L/r^2}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{L^2}{2\mu r^2}\right)}} \end{aligned}$$

where we have chosen the positive square root. Now the term in the denominator under the square root is a quadratic in $1/r$, so completing the square here (see **(a)**) gives:

$$\begin{aligned} -\frac{L^2}{2\mu}\left(\frac{1}{r}\right)^2 + k\left(\frac{1}{r}\right) + E &= \frac{\mu k^2}{2L^2} + E - \frac{L^2}{2\mu}\left(\frac{1}{r} - \frac{\mu k}{L^2}\right)^2 \\ &= \frac{\mu k^2}{2L^2}\left(1 + \frac{2EL^2}{\mu k^2} - \left(\frac{L^2}{\mu k}\right)^2\left(\frac{1}{r} - \frac{\mu k}{L^2}\right)^2\right) \\ &= \frac{\mu k^2}{2L^2}\left(1 + \frac{2EL^2}{\mu k^2} - \left(\frac{L^2}{\mu k r} - 1\right)^2\right) \end{aligned}$$

so that

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = \frac{L/r^2}{\sqrt{\frac{\mu^2k^2}{L^2}\left(1+\frac{2EL^2}{\mu k^2}-\left(\frac{L^2}{\mu kr}-1\right)^2\right)}}$$
$$= \frac{L^2/\mu k}{r^2\sqrt{\left(1+\frac{2EL^2}{\mu k^2}-\left(\frac{L^2}{\mu kr}-1\right)^2\right)}}$$

Now, let $\alpha = L^2/\mu k$ and $\varepsilon^2 = 1+\frac{2EL^2}{\mu k^2}$ and, with reference to the sketch in part **(a)**, choose $\theta = 0$ corresponding to $r = \alpha/(1+\varepsilon)$. Hence

$$\int_0^\theta \mathrm{d}\theta = \int_{\frac{\alpha}{1+\varepsilon}}^r \mathrm{d}r \frac{\alpha}{r^2\sqrt{\left(\varepsilon^2-\left(\frac{\alpha}{r}-1\right)^2\right)}}$$

Change variables to $u = \frac{\alpha}{r}-1$ so that $\mathrm{d}u = -\frac{\alpha}{r^2}\mathrm{d}r$ which leads to

$$\theta = \int_\varepsilon^{\frac{\alpha}{r}-1} \mathrm{d}u \frac{-1}{\sqrt{(\varepsilon^2-u^2)}}$$

This is the standard integral quoted, or, proceeding explicitly, change variables again to $u = \varepsilon\cos w$ so that $\mathrm{d}u = -\varepsilon\sin w\,\mathrm{d}w$ leading to

$$\theta = \int_0^{\arccos((\alpha/r-1)/\varepsilon)} \frac{\mathrm{d}w\,\sin w}{\sqrt{(1-\cos^2 w)}}$$
$$= \int_0^{\arccos((\alpha/r-1)/\varepsilon)} \mathrm{d}w$$
$$= \arccos\left(\frac{1}{\varepsilon}\left(\frac{\alpha}{r}-1\right)\right)$$

Max 4 for progress in this section

giving $\frac{\alpha}{r} = 1+\varepsilon\cos\theta$ and hence

$$\boxed{r = \frac{\alpha}{1+\varepsilon\cos\theta}}$$

Note that the definition of α and ε, together with the fact that $\alpha = a(1-\varepsilon^2)$ means that $E = -\frac{k}{2a}$.

(iii) Go back to Kepler's second law

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{L}{2m_p}$$

Over a time interval equal to the orbital period T this implies that

$$A = \frac{L}{2m_p}T$$

where A is the area of the ellipse.

The area of an ellipse is given in the question, but the calculation is straight-forward:
The area of an ellipse is most easily found from the cartesian form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

so

$$\begin{aligned} A &= 2\int_{-a}^{a} y\,\mathrm{d}x \\ &= 2\int_{-a}^{a} \sqrt{b^2 - \frac{b^2}{a^2}x^2}\,\mathrm{d}x \\ &= \frac{b}{a}2\int_{-a}^{a} \sqrt{a^2 - x^2}\,\mathrm{d}x \end{aligned}$$

but this is just

$$\begin{aligned} A &= \frac{b}{a} \times A_{\text{circle radius } a} \\ &= \frac{b}{a}\pi a^2 \\ &= \pi ab \end{aligned}$$

So

$$T = \frac{2m_p \pi ab}{L}$$

Now, from the relation between a, b and ε, we have

$$b^2 = a^2(1 - \varepsilon^2)$$

and from the definition of α

$$L^2 = m_p k\alpha$$

and finally from the connection between α and a

$$\alpha = a(1 - \varepsilon^2)$$

Putting all this together

$$\begin{aligned} T^2 &= \frac{4m_p^2\pi^2 a^2 b^2}{L^2} \\ &= \frac{4m_p^2\pi^2 a^4(1-\varepsilon^2)}{m_p k\alpha} \\ &= \frac{4m_p\pi^2 a^4(1-\varepsilon^2)}{ka(1-\varepsilon^2)} \\ &= \frac{4\pi^2 m_p}{k} a^3 \end{aligned}$$

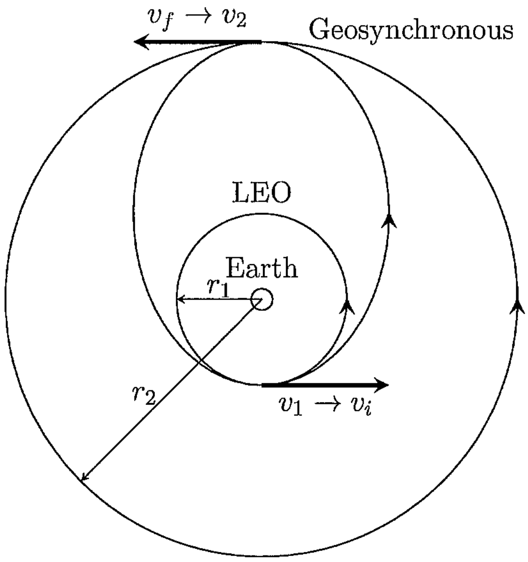
Max 4 marks

that is

$$\boxed{T^2 = \frac{4\pi^2}{GM_S} a^3}$$

Note: In parts **(c)** and **(d)**, the more general μ has been used in place of m_p to start with, with the simplification $M_S + m_p \approx M_S$ and $\mu \approx m_p$ taken at the end. Note that in terms of these, the period of orbit is given by $T^2 = \frac{4\pi^2\mu}{k}a^3 = \frac{4\pi^2}{G(M_S+m_p)}$.

(c) Assuming that the low earth orbit (LEO) and geosynchronous orbits are circular to a high degree of accuracy (as they usually are), and that they lie in the same plane, and that the impulses occur effectively instantaneously, the situation is as follows



and all of the formulas previously arrived at hold with the replacements $M_S \to M_E$ and $m_p \to m_s$ with M_E the mass of Earth and m_s the mass of the satellite. Furthermore, since $m_s/M_E \sim \mathcal{O}(10^{-21})$, $M_E \gg m_s$ and we may approximate $\mu \approx m_s$.

With the radius of Earth, $r_E = 6380\,\text{km}$,

$$r_1 = r_E + h = 6.38 \times 10^6\,\text{m} + 0.32 \times 10^6\,\text{m} = 6.70 \times 10^6\,\text{m}$$

while the radius of a geosynchronous orbit may be found from Kepler's third law

$$\begin{aligned} r_2 = a = \left(\frac{G(M_E+m_s)T^2}{4\pi^2}\right)^{\frac{1}{3}} &\approx \left(\frac{GM_ET^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &= \left(\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times (24 \times 3600)^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &= 42.23 \times 10^6\,\text{m} \end{aligned}$$

Now since r_1 is the radius at perihelion and r_2 is the radius at aphelion of the transfer orbit we have (see sketch in part **(a)**)

$$\begin{aligned} r_1 &= \frac{\alpha}{1+\varepsilon} = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon) \\ r_2 &= \frac{\alpha}{1-\varepsilon} = \frac{a(1-\varepsilon^2)}{1-\varepsilon} = a(1+\varepsilon) \end{aligned}$$

One mark per significant step they have made
Maximum of 20 marks for the question

so $r_2 - r_1 = 2a\varepsilon$ and

$$\varepsilon = \frac{r_2 - r_1}{2a} = \frac{r_2 - r_1}{r_2 + r_1}$$

This gives the transfer orbit an eccentricity of

$$\boxed{\varepsilon = \frac{42.23 - 6.70}{42.23 + 6.70} = 0.726}$$

i.e. it is a highly eccentric orbit.

Now the velocities of the circular orbits may be found from Newton's second law via

$$\begin{aligned} \frac{\mu v_1^2}{r_1} &= \frac{k}{r_1^2} \\ \Rightarrow v_1^2 &= \frac{k}{\mu r_1} \end{aligned}$$

and similarly $v_2^2 = \frac{k}{\mu r_2}$ (we will soon approximate $\mu \approx m_s$). For v_i and v_f, the velocities at periapsis and apoapsis in the elliptical orbit, go back to conservation of energy and recall (from **(b)**) that $E = -k/2a$. So

$$\begin{aligned} \frac{1}{2}\mu v_i^2 - \frac{k}{r_1} &= -\frac{k}{r_1 + r_2} \\ \Rightarrow v_i^2 &= 2\frac{k}{\mu}\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right) \\ &= \frac{k}{\mu r_1}\left(\frac{2r_2}{r_1 + r_2}\right) \end{aligned}$$

and similarly $v_f^2 = \frac{k}{\mu r_2}\left(\frac{2r_1}{r_1+r_2}\right)$. Summarising

$$\begin{aligned} v_1^2 &= \frac{k}{\mu r_1} \\ v_2^2 &= \frac{k}{\mu r_2} \\ v_i^2 &= \frac{k}{\mu r_1}\left(\frac{2r_2}{r_1 + r_2}\right) \\ v_f^2 &= \frac{k}{\mu r_2}\left(\frac{2r_1}{r_1 + r_2}\right) \end{aligned}$$

so that at periapsis, $\Delta v_p = v_i - v_1$ is given by

$$\Delta v_p = \sqrt{\frac{k}{\mu r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

and at apoapsis, $\Delta v_a = v_2 - v_f$ is

$$\Delta v_a = \sqrt{\frac{k}{\mu r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

Finally, setting $\mu \approx m_s$ and $k = GM_E m_s$ gives

$$\begin{aligned}
\Delta v_p &= \sqrt{\frac{GM_E}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right) \\
\Delta v_a &= \sqrt{\frac{GM_E}{r_2}}\left(1-\sqrt{\frac{2r_1}{r_1+r_2}}\right)
\end{aligned}$$

which then give

$$\begin{aligned}
\Delta v_p &= \sqrt{\frac{6.67\times10^{-11}\times5.97\times10^{24}}{6.70\times10^{6}}}\left(\sqrt{\frac{2\times42.23}{6.70+42.23}}-1\right) \\
&= \underline{2.42\,\mathrm{kms}^{-1}}
\end{aligned}$$

and

$$\begin{aligned}
\Delta v_a &= \sqrt{\frac{6.67\times10^{-11}\times5.97\times10^{24}}{42.23\times10^{6}}}\left(1-\sqrt{\frac{2\times6.70}{6.70+42.23}}\right) \\
&= \underline{1.46\,\mathrm{kms}^{-1}}
\end{aligned}$$

Note that the velocity of the craft is *increased* at both points giving a total Δv of

$$\begin{aligned}
\Delta v &= |\Delta v_p| + |\Delta v_a| \\
&= 2.42\times10^{3}\,\mathrm{ms}^{-1} + 1.46\times10^{3}\,\mathrm{ms}^{-1} \\
&= \underline{3.88\,\mathrm{kms}^{-1}}
\end{aligned}$$

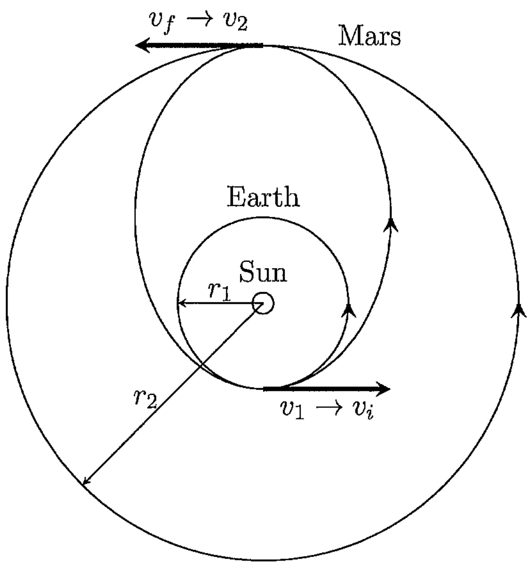
The time taken to achieve transfer is simply half the time period of the transfer orbit

$$\begin{aligned}
t_{\mathrm{transfer}} &= \frac{1}{2}T \\
&= \frac{1}{2}\sqrt{\frac{4\pi^2a^3}{G(M_E+m_s)}} \\
&= \frac{1}{2}\sqrt{\frac{\pi^2(r_1+r_2)^3}{2G(M_E+m_s)}} \\
&\approx \frac{1}{2}\sqrt{\frac{\pi^2(r_1+r_2)^3}{2GM_E}} \\
&= \frac{1}{2}\sqrt{\frac{\pi^2\times(6.70+42.23)^3\times10^{18}}{2\times6.67\times10^{-11}\times5.97\times10^{24}}} \\
&= \underline{19050\,\mathrm{s} = 318\,\mathrm{min} = 5.29\,\mathrm{hr}}
\end{aligned}$$

(d) The concepts regarding the transfer orbit from Earth's orbit around the Sun to Mars' orbit will be the same as in part **(c)** giving

$$\begin{aligned}
v_1^2 &= \frac{k}{\mu r_1} \\
v_2^2 &= \frac{k}{\mu r_2} \\
v_i^2 &= \frac{k}{\mu r_1}\left(\frac{2r_2}{r_1+r_2}\right) \\
v_f^2 &= \frac{k}{\mu r_2}\left(\frac{2r_1}{r_1+r_2}\right)
\end{aligned}$$

similarly to in **(d)**, but where $k = GM_S m_s$ and $\mu = \frac{M_S m_s}{M_S+m_s}$.



However, it will also be necessary to transfer the craft from its orbit around the Earth to its transfer orbit at the start and from its transfer orbit to orbit around Mars at the end. Taking the departure from Earth first, the velocity necessary to reach for the transfer orbit is v_i. However, this is in a frame of reference relative to the Sun. In a frame of reference relative to the Earth the required velocity is $V_i = v_i - v_1$, but this is just Δv_p as defined in part **(d)** with the new definition of μ and k. If the spacecraft is originally in orbit around the Earth at radius R_1 (use capital letters for orbits around planets in this part of the question, and lowercase letters for orbits of planets around the Sun), then a velocity V_{escape} must be reached from that orbit which from energy conservation can be found from

$$\begin{aligned}
\frac{1}{2}\mu_1 V_i^2 &= \frac{1}{2}\mu_1 V_{\text{escape}}^2 - \frac{k_1}{R_1} \\
\Rightarrow V_{\text{escape}}^2 &= V_i^2 + \frac{2k_1}{\mu_1 R_1}
\end{aligned}$$

where $k_1 = GM_E m_s$ and $\mu_1 = \frac{M_E m_s}{M_E+m_s}$ as in **(c)**. There is no potential energy term on the left hand side since the spacecraft is assumed to have left the sphere of influence of the Earth once it enters the transfer orbit. However, since the spacecraft was initially in a circular orbit of radius R_1 around the Earth with velocity (from Newton's second law for circular orbits)

$$V_1^2 = \frac{k_1}{\mu_1 R_1}$$

so the initial Δv required is

$$\begin{aligned}
\Delta v_p' &= V_{\text{escape}} - V_1 \\
&= \sqrt{V_i^2 + \frac{2k_1}{\mu_1 R_1}} - \sqrt{\frac{k_1}{\mu_1 R_1}} \\
&= \sqrt{\frac{k}{\mu r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)^2 + \frac{2k_1}{\mu_1 R_1}} - \sqrt{\frac{k_1}{\mu_1 R_1}}
\end{aligned}$$

Finally, approximating $\mu \approx m_s$ and $\mu_1 \approx m_s$, and substituting $k_1 = GM_E m_s$ and $k = GM_S m_s$ we have

$$\Delta v_p' = \sqrt{\frac{GM_S}{r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)^2 + \frac{2GM_E}{R_1}} - \sqrt{\frac{GM_E}{R_1}}$$

Likewise, on arrival at Mars the spacecraft has velocity v_f relative to the Sun. However as Mars is travelling at velocity v_1 in the same direction (with $v_1 > v_f$), its velocity relative to Mars is $V_f = v_f - v_1$. A velocity V_{capture} must be reached from that for which (here $k_2 = GM_M m_s$ and $\mu_2 = \frac{M_M m_s}{M_M + m_s}$)

$$\begin{aligned}
\frac{1}{2}\mu_2 V_f^2 &= \frac{1}{2}\mu_2 V_{\text{capture}}^2 - \frac{k_2}{R_2} \\
\Rightarrow V_{\text{capture}}^2 &= V_f^2 + \frac{2k_2}{\mu_2 R_2}
\end{aligned}$$

and with orbital velocity

$$V_2^2 = \frac{k_2}{\mu_2 R_2}$$

giving a final Δv of

$$\begin{aligned}
\Delta v_a' &= V_{\text{capture}} - V_2 \\
&= \sqrt{V_f^2 + \frac{2k_2}{\mu_2 R_2}} - \sqrt{\frac{k_2}{\mu_2 R_2}} \\
&= \sqrt{\frac{k}{\mu r_2}\left(\sqrt{\frac{2r_1}{r_1+r_2}} - 1\right)^2 + \frac{2k_2}{\mu_2 R_2}} - \sqrt{\frac{k_2}{\mu_2 R_2}}
\end{aligned}$$

Finally, approximating $\mu \approx m_s$ and $\mu_2 \approx m_s$, and substituting $k_2 = GM_M m_s$ and $k = GM_S m_s$ we have

$$\Delta v_a' = \sqrt{\frac{GM_S}{r_2}\left(\sqrt{\frac{2r_1}{r_1+r_2}} - 1\right)^2 + \frac{2GM_M}{R_2}} - \sqrt{\frac{GM_M}{R_2}}$$

So the total Δv budget will arise from

$$\boxed{\begin{aligned} \Delta v'_p &= \sqrt{\frac{GM_S}{r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)^2+\frac{2GM_E}{R_1}}-\sqrt{\frac{GM_E}{R_1}} \\ \Delta v'_a &= \sqrt{\frac{GM_S}{r_2}\left(\sqrt{\frac{2r_1}{r_1+r_2}}-1\right)^2+\frac{2GM_M}{R_2}}-\sqrt{\frac{GM_M}{R_2}} \end{aligned}}$$

Taking 365 days to orbit the Sun, the radius of the Earth's orbit about the Sun is approximately:

$$\begin{aligned} r_1 &= \left(\frac{G(M_S+M_E)T_E^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &\approx \left(\frac{GM_ST_E^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &= \left(\frac{6.67\times10^{-11}\times1.99\times10^{30}\times(365\times24\times3600)^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &\approx 1.50\times10^{11}\,\mathrm{m} \end{aligned}$$

Likewise for Mars

$$\begin{aligned} r_2 &= \left(\frac{G(M_S+M_M)T_M^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &\approx \left(\frac{GM_ST_M^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &= \left(\frac{6.67\times10^{-11}\times1.99\times10^{30}\times(1.88\times365\times24\times3600)^2}{4\pi^2}\right)^{\frac{1}{3}} \\ &\approx 2.28\times10^{11}\,\mathrm{m} \end{aligned}$$

and

$$\begin{aligned} R_1 &= r_E+300\,\mathrm{km} \\ &= 6.38\times10^6\,\mathrm{m}+0.3\times10^6\,\mathrm{m} \\ &= 6.68\times10^6\,\mathrm{m} \end{aligned}$$

with

$$\begin{aligned} R_2 &= r_M+250\,\mathrm{km} \\ &= 3.40\times10^6\,\mathrm{m}+0.25\times10^6\,\mathrm{m} \\ &= 3.65\times10^6\,\mathrm{m} \end{aligned}$$

Giving

$$\begin{aligned} \Delta v'_p &= \sqrt{\frac{GM_S}{r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)^2+\frac{2GM_E}{R_1}}-\sqrt{\frac{GM_E}{R_1}} \\ &= \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{1.50\times10^{11}}\left(\sqrt{\frac{2\times2.28}{1.50+2.28}}-1\right)^2+\frac{2\times6.67\times10^{-11}\times5.97\times10^{24}}{6.68\times10^6}}-\sqrt{\frac{6.67\times10^{-11}\times5.97\times10^{24}}{6.68\times10^6}} \\ &= \underline{3.58\,\mathrm{kms}^{-1}} \end{aligned}$$

and

$$\begin{aligned}
\Delta v_a' &= \sqrt{\frac{GM_S}{r_2}\left(\sqrt{\frac{2r_1}{r_1+r_2}}-1\right)^2+\frac{2GM_M}{R_2}}-\sqrt{\frac{GM_M}{R_2}} \\
&= \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{2.28\times10^{11}}\left(\sqrt{\frac{2\times1.50}{1.50+2.28}}-1\right)^2+\frac{2\times6.67\times10^{-11}\times6.42\times10^{23}}{3.65\times10^{6}}}-\sqrt{\frac{6.67\times10^{-11}\times6.42\times10^{23}}{3.65\times10^{6}}} \\
&= \underline{2.09\,\mathrm{kms}^{-1}}
\end{aligned}$$

The total Δv budget is therefore

$$\begin{aligned}
\Delta v &= |\Delta v_p'|+|\Delta v_a'| \\
&= 3.58\,\mathrm{kms}^{-1}+2.09\,\mathrm{kms}^{-1} \\
&= \underline{5.67\,\mathrm{kms}^{-1}}
\end{aligned}$$

In calculating this we have assumed that the fuel burn to generate $\Delta v_p'$ takes place all at once and effectively instantaneously. Likewise the $\Delta v_a'$ burn. Furthermore the orbits of Earth and Mars are taken as being circular. This is not such a bad approximation for Earth, with its eccentricity of 0.017, but is not such a good approximation for Mars, which has the second most eccentric elliptical orbit in the solar system after Mercury. Furthermore, the eccentricity of the transfer orbit is

$$\begin{aligned}
\varepsilon &= \frac{r_2-r_1}{r_2+r_1} \\
&= \frac{2.28-1.50}{2.28+1.50} \\
&\approx 0.206
\end{aligned}$$

which is of the same order of magnitude of the eccentricity of Mars, so a treatment of the orbit of Mars as circular is not really justified. Mars' position at the instant of proximity would therefore have to be more accurately calculated. On a related note, the plane of the orbit of Mars is inclined to the plane of the orbit of Earth (at an angle of almost 2°) and this would need to be accounted for too.

The time taken for the transfer to Mars would simply be half the period of the transfer orbit:

$$\begin{aligned}
t_{\text{transfer}} &= \frac{1}{2}T \\
&= \frac{1}{2}\sqrt{\frac{4\pi^2a^3}{G(M_S+m_s)}} \\
&= \frac{1}{2}\sqrt{\frac{\pi^2(r_1+r_2)^3}{2G(M_S+m_s)}} \\
&\approx \frac{1}{2}\sqrt{\frac{\pi^2(r_1+r_2)^3}{2GM_S}} \\
&= \frac{1}{2}\sqrt{\frac{\pi^2\times(1.50+2.28)^3\times10^{33}}{2\times6.67\times10^{-11}\times1.99\times10^{30}}} \\
&= \underline{2.24\times10^7\,\mathrm{s}=259\text{ days}}
\end{aligned}$$

Now

$$\begin{aligned} r_E\omega_E &= \sqrt{\frac{GM_S}{r_1}} \\ \Rightarrow \omega_E &= \sqrt{\frac{GM_S}{r_1^3}} \end{aligned}$$

and likewise

$$\omega_M = \sqrt{\frac{GM_S}{r_2^3}}$$

Since Mars will travel through an angle of $\omega_M t_{\text{transfer}}$ between the time of launch of the spacecraft from Earth and the time of its arrival at Mars, the initial angle between Earth and Mars at the time of launch will be

$$\theta_0 = \pi - \omega_M t_{\text{transfer}}$$

So, choosing $t = 0$ to be the moment of departure of the spacecraft from Earth, and referring everything to the position of Earth at this instant

$$\begin{aligned} \theta_E &= \omega_E t \\ \theta_M &= \omega_M t + \theta_0 \end{aligned}$$

Then the angle between a radius joining the Sun with Earth and the Sun with Mars is

$$\begin{aligned} \theta_{ME} &= \theta_M - \theta_E \\ &= (\omega_M - \omega_E)t + \theta_0 \end{aligned}$$

Now since $\omega_E > \omega_M$, θ_{ME} will start from θ_0 and will rapidly decrease before becoming negative. In particular, when the craft arrives at Mars, $t = t_{\text{transfer}}$ and $\theta_{ME} = \theta_f$ where

$$\begin{aligned} \theta_f &= (\omega_M - \omega_E)t_{\text{transfer}} + \pi - \omega_M t_{\text{transfer}} \\ &= \pi - \omega_E t_{\text{transfer}} \end{aligned}$$

Redefining $t = 0$ to be the time of arrival for the return journey (and calling the newly defined angle θ'_{ME})

$$\theta'_{ME} = (\omega_M - \omega_E)t + \theta_f$$

But, by the symmetry of the situation, when departing from Mars for Earth, the angle between Earth and Mars must be $\theta'_0 = \pi - \omega_E t_{\text{transfer}}$ and we wish to know what $t = t_{\text{waiting}}$ is for $\theta'_{ME} = -\theta'_0$ (minus since Earth will have overtaken Mars by this point). But, $\theta'_0 = \theta_f$ so

$$t_{\text{waiting}} = \frac{-2\theta_f}{\omega_M - \omega_E}$$

This equation may produce a negative value - indicating that the required time actually occurs before the craft has arrived at Mars - and this is of course because $-2\theta_f$ is only defined up to 2π. In general, therefore

$$\boxed{t_{\text{waiting}} = \frac{-2\theta_f - 2\pi n}{\omega_M - \omega_E}}$$

or

$$\boxed{t_{\text{waiting}} = \frac{-2\theta_f - 2\pi n}{\sqrt{GM_S}\left(\frac{1}{r_2^{3/2}} - \frac{1}{r_1^{3/2}}\right)}}$$

For $n = 0$ this gives

$$\begin{aligned}
t_{\text{waiting}} &= \frac{-2 \times (\pi - \frac{\sqrt{GM_S}}{r_1^{3/2}} t_{\text{transfer}})}{\sqrt{GM_S}\left(\frac{1}{r_2^{3/2}} - \frac{1}{r_1^{3/2}}\right)} \\
&= \frac{-2 \times \left(\pi - \frac{\sqrt{6.67\times10^{-11}\times1.99\times10^{30}}}{(1.50\times10^{11})^{3/2}} \times 2.24 \times 10^7\right)}{\sqrt{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}\left(\frac{1}{(2.28\times10^{11})^{3/2}} - \frac{1}{(1.50\times10^{11})^{3/2}}\right)} \\
&= -2.81 \times 10^7\,\text{s}
\end{aligned}$$

This is negative so we must go to at least $n = 1$:

$$\begin{aligned}
t_{\text{waiting}} &= \frac{-2 \times (\pi - \frac{\sqrt{GM_S}}{r_1^{3/2}} t_{\text{transfer}}) - 2\pi}{\sqrt{GM_S}\left(\frac{1}{r_2^{3/2}} - \frac{1}{r_1^{3/2}}\right)} \\
&= \frac{-2 \times \left(\pi - \frac{\sqrt{6.67\times10^{-11}\times1.99\times10^{30}}}{(1.50\times10^{11})^{3/2}} \times 2.24 \times 10^7\right) - 2\pi}{\sqrt{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}\left(\frac{1}{(2.28\times10^{11})^{3/2}} - \frac{1}{(1.50\times10^{11})^{3/2}}\right)} \\
&= 3.98 \times 10^7\,\text{s} = 460\,\text{days}
\end{aligned}$$

Giving ultimately

$$\begin{aligned}
t_{\text{trip}} &= 2t_{\text{transfer}} + t_{\text{waiting}} \\
&= (2 \times 259 + 460)\,\text{days} = \underline{978\,\text{days} = 2.7\,\text{years} \approx 2\,\text{yrs}\ 8\,\text{months}}
\end{aligned}$$

Qu5. Thermal Properties of Ideal Gases

(a) A monatomic ideal gas has $\frac{3}{2}kT$ of kinetic energy per molecule on average. There are no further contributions to the energy from rotational/vibrational modes, so the internal energy is:

$$U = \frac{3}{2}nRT$$

and hence $\delta U = \frac{3}{2}nR\delta T$. A monatomic ideal gas has equation of state $pV = nRT$, and furthermore the first law of thermodynamics states that

$$\delta U = \delta Q + \delta W$$

where δQ is the heat supplied to the system and $\delta W = -p\delta V$ is the work done on the system.

(I) The definition of heat capacity at constant volume, C_V is:

$$\delta Q = C_V \delta T$$

Since the volume of gas is constant, $\delta V = 0$ so $\delta W = 0$. The first law therefore reduces to $\delta U = \delta Q$ so that $\frac{3}{2}nR\delta T = C_V\delta T$ and hence

$$\boxed{C_V = \frac{3}{2}nR}$$

(II) The definition of heat capacity at constant pressure, C_p is:

$$\delta Q = C_p \delta T$$

Substituting this into the first law gives

$$\underbrace{\frac{3}{2}nR}_{C_V}\delta T = C_p\delta T - p\delta V \tag{3}$$

but using the equation of state $pV = nRT$ implies

$$p\delta V + V\delta p = nR\delta T$$

and since pressure is constant, $\delta p = 0$ so $p\delta V = nR\delta T$. This means that (3) becomes

$$C_V\delta T = C_p\delta T - nR\delta T$$

i.e.

$$\boxed{C_p = C_V + nR}$$

or

$$\boxed{C_p = \frac{5}{2}nR}$$

for an ideal gas.

4 marks.

(b) Note: Initial derivation here not needed as part of answer, of course.

An adiabatic change has $\delta Q = 0$, so the first law becomes $\delta U = \delta W$, or $\frac{3}{2}nR\delta T = -p\delta V$. But using the equation of state as in **(II)**, $p\delta V + V\delta p = nR\delta T$, so

$$\begin{aligned} \frac{C_V}{nR}(p\delta V + V\delta p) &= -p\delta V \\ \Rightarrow \frac{C_V}{nR}V\delta p &= -\left(1+\frac{C_V}{nR}\right)p\delta V \\ \Rightarrow \frac{\delta p}{p} &= -\left(\frac{C_V+nR}{C_V}\right)\frac{\delta V}{V} \\ \Rightarrow \frac{\delta p}{p} &= -\left(\frac{C_p}{C_V}\right)\frac{\delta V}{V} \end{aligned}$$

Let $\frac{C_p}{C_V} = \gamma$, then

$$\begin{aligned} \int \frac{\mathrm{d}p}{p} &= -\gamma \int \frac{\mathrm{d}V}{V} \\ \Rightarrow \ln p &= -\gamma \ln V + \text{const.} \\ \Rightarrow \ln p &= \ln\left(\text{const.}V^{-\gamma}\right) \end{aligned}$$

Not required

so that

Given $\boxed{pV^\gamma = \text{const.}}$

with $\gamma = \frac{C_p}{C_V} = \frac{5}{3}$. Then using the equation of state, $pV = nRT$, $p\left(\frac{nRT}{p}\right)^\gamma = \text{const.}$ so

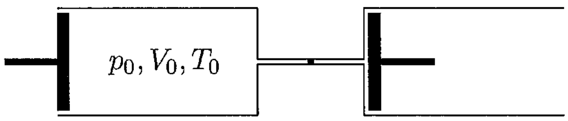
$$\boxed{p^{1-\gamma}T^\gamma = \text{const.}}$$

and $\frac{nRT}{V}V^\gamma = \text{const.}$ so

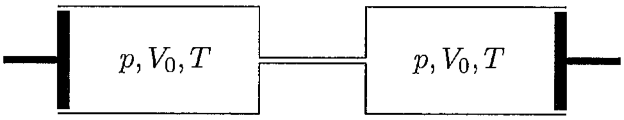
$$\boxed{TV^{\gamma-1} = \text{const.}}$$

2

(c) Starting configuration



(i) Final configuration



Process performed slowly and with cylinders thermally isolated so process is **adiabatic.** Hence (*e.g.*) $TV^{\gamma-1} = \text{const}$. Therefore:

$$T_0 V_0^{\gamma-1} = T(2V_0)^{\gamma-1}$$

so

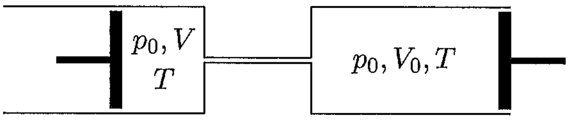
$$T = 2^{1-\gamma} T_0$$

A monatomic gas has $\gamma = 5/3$ so

$$\boxed{T = 2^{-2/3} T_0 \approx 0.63 T_0}$$

3

(ii) Final configuration



Let $V = \alpha V_0$. Conservation of number of moles of gas between start and end gives:

$$n = \frac{p_0 V_0}{T_0} = \frac{p_0(V_0 + \alpha V_0)}{T}$$

so

$$\frac{T}{T_0} = 1 + \alpha \tag{4}$$

When valve is 'slightly' opened, no work is done on gas but it naturally expands to fill volume $2V_0$. No heat is exchanged as system is thermally isolated, so $\delta Q = 0$, hence $\delta U = 0$ implying $\delta T = 0$ for an ideal gas – its temperature remains at T_0 for this part of the process. Pressure remains at p_0.

Work done on gas during motion of piston is then

$$\delta W = -p_0 \delta V = -p_0(V_0 + \alpha V_0 - 2V_0) = -p_0 V_0(\alpha - 1)$$

As cylinders are thermally isolated from the surroundings, no heat is exchanged so motion of piston is adiabatic with $\delta W = \delta U = \frac{3}{2} nR\delta T$ so

$$-p_0 V_0(\alpha - 1) = \frac{3}{2} nR(T - T_0)$$

or, rearranging

$$\frac{p_0 V_0}{T_0}(1 - \alpha) = \frac{3}{2} nR \left(\frac{T}{T_0}\right)$$

But, using the ideal gas law, $p_0 V_0 / T_0 = nR$, giving

$$\frac{3}{2}\left(\frac{T}{T_0}\right) = (1 - \alpha) \tag{5}$$

Adding (4) and (5) gives

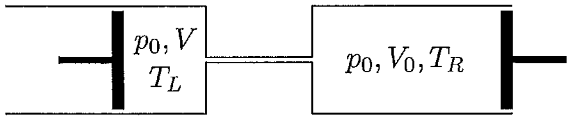
$$\frac{5}{2}\frac{T}{T_0} - \frac{3}{2} = 2$$

so

$$\boxed{T = \frac{7}{5}T_0}$$

5

(iii) Final configuration



Now the cylinders are kept thermally isolated from each other so their final temperatures may be different. Let $V = \beta V_0$. The work done on the gas is (as in **(ii)**):

$$\delta W = -p\delta V = p_0 V_0 (1 - \beta)$$

and since no heat is exchanged, $\delta W = \delta U$, though the change in internal energy is different to before. n_L moles change in temperature by $T_L - T_0$, while n_R moles change in temperature by $T_R - T_0$ so that

$$\begin{aligned}
\delta U &= \frac{3}{2}R\Big(n_L(T_L - T_0) + n_R(T_R - T_0)\Big) \\
&= \frac{3}{2}\Big(n_L R T_L - n_L R T_0 + n_R R T_R - n_R R T_0\Big) \\
&= \frac{3}{2}\Big(n_L R T_L + n_R R T_R - (n_L + n_R) R T_0\Big)
\end{aligned}$$

Using the ideal gas law (and the fact that pressure is kept constant), $n_L R T_L = p_0 \beta V_0$, $n_R R T_R = p_0 V_0$ and $(n_L + n_R) R T_0 = n_0 R T_0 = p_0 V_0$, meaning that δU reduces to

$$\begin{aligned}
\delta U &= \frac{3}{2}(\beta p_0 V_0 + p_0 V_0 - p_0 V_0) \\
&= \frac{3}{2}\beta p_0 V_0
\end{aligned}$$

The first law then becomes

$$\begin{aligned}
\delta W &= \delta U \\
\Rightarrow p_0 V_0 (1 - \beta) &= \frac{3}{2}\beta p_0 V_0
\end{aligned}$$

giving

$$\beta = \frac{2}{5}$$

Because the cylinders are thermally isolated from each other, only the gas in the left cylinder undergoes an adiabatic change, and for the gas in the left cylinder, therefore, $p^{1-\gamma}T^{\gamma}$ = const.. Since pressure is constant, temperature of the gas in the left cylinder must also be constant, *i.e.* $\boxed{T_L = T_0}$.
Using the ideal gas law for the gas in the right cylinder, $T_R = \frac{p_0 V_0}{n_R R}$, and n_R can be found from

$$\begin{aligned} n_R &= n_0 - n_L \\ &= \frac{p_0 V_0}{RT_0} - \frac{p_0 \beta V_0}{RT_L} \\ &= \frac{p_0 V_0}{RT_0} - \frac{p_0 \beta V_0}{RT_0} \\ &= (1-\beta)\frac{p_0 V_0}{RT_0} \end{aligned}$$

so

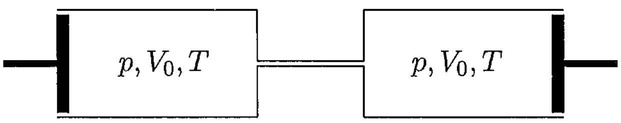
$$\begin{aligned} T_R &= \frac{p_0 V_0}{n_R R} \\ &= \frac{p_0 V_0}{(1-\beta)p_0 V_0/(RT_0)} \\ &= \frac{T_0}{1-\beta} \end{aligned}$$

giving $\boxed{T_R = \frac{5}{3}T_0}$. Overall, then:

$$\boxed{T_L = T_0 \qquad T_R = \frac{5}{3}T_0}$$

4

(iv) Final configuration

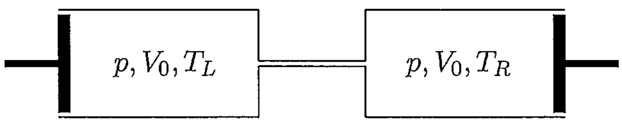


No work is done, since the gas is simply allowed to expand, and no heat is exchanged since the cylinders are thermally isolated. This is a joule expansion. Since $\delta W = \delta Q = 0$, $\delta U = 0$, and since it is an ideal gas this means that $\delta T = 0$ too. Hence:

$$\boxed{T = T_0}$$

1

(v) Final configuration



The internal energy of an ideal gas is (of course) $U = \frac{3}{2}nRT$, so conservation of energy implies

$$\begin{aligned} \frac{3}{2}n_0RT_0 &= \frac{3}{2}n_LRT_L + \frac{3}{2}n_RRT_R \\ \Rightarrow n_0RT_0 &= n_LRT_L + n_RRT_R \end{aligned}$$

and using the equation of state for an ideal gas, $pV = nRT$ gives

$$p_0V_0 = pV_0 + pV_0$$

that is

$$p = \frac{p_0}{2}$$

Conservation of number of moles of gas then means

$$\begin{aligned} n_0 &= n_L + n_R \\ \Rightarrow \frac{p_0V_0}{RT_0} &= \frac{pV_0}{RT_L} + \frac{pV_0}{RT_R} \\ \Rightarrow \frac{p_0V_0}{RT_0} &= \frac{p_0V_0}{2RT_L} + \frac{p_0V_0}{2RT_R} \end{aligned}$$

so

$$\frac{1}{T_0} = \frac{1}{2}\left(\frac{1}{T_L} + \frac{1}{T_R}\right) \tag{6}$$

Again, the gas in the left cylinder undergoes an adiabatic change so obeys $p^{1-\gamma}T^{\gamma} = \text{const.}$ Raising this to the power of $1/\gamma$ gives

$$p^{\frac{1-\gamma}{\gamma}}T = \text{const.}$$

Recalling that $\gamma = \frac{5}{3}$ for an ideal monatomic gas, $\frac{1-\gamma}{\gamma} = -\frac{2}{5}$ so

$$\begin{aligned} \frac{T_L}{T_0} &= \left(\frac{p}{p_0}\right)^{\frac{1-\gamma}{\gamma}} \\ &= \left(\frac{1}{2}\right)^{2/5} \end{aligned}$$

so

$$\boxed{T_L = 2^{-2/5}T_0 \approx 0.76T_0}$$

Finally, from (6)

$$\begin{aligned} T_R &= \frac{T_0T_L}{2T_L - T_0} \\ &= \frac{2^{-2/5}}{2^{3/5} - 1}T_0 \end{aligned}$$

so

$$\boxed{T_R = \frac{1}{2 - 2^{2/5}}T_0 \approx 1.47T_0}$$

4

Maximum of 20