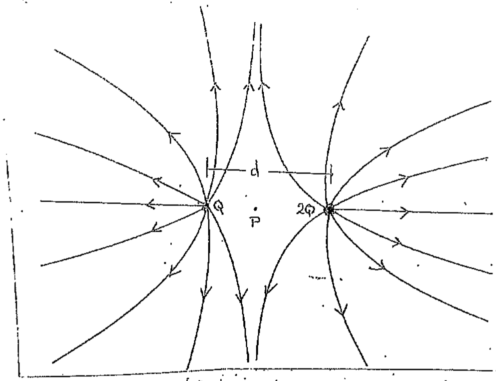
2017 BPhO Round 1 Solutions

Q1 (a)



Qualitatively correct graph with arrows and symmetry about x-axis. 2

(i) Neutral point P at distance r_1 from +Q and r_2 from +2Q.

So

$$\frac{Q}{4\pi\varepsilon_0 r_1^2} = \frac{2Q}{4\pi\varepsilon_0 r_2^2}$$

$$\underline{r_2 = \sqrt{2}\, r_1} \qquad (1) \qquad 1$$

Now $r_1 + r_2 = d$

So $r_1(1+\sqrt{2}) = d$

Giving $r_1 = 0.414\,d$ } either result 1

From (1) $r_2 = 0.586\,d$

(ii) Field lines are defined as being tangential to the field vector at all points along the line. Consequently the magnitude varies or The magnitude varies with the density of the field lines (1) 1

5

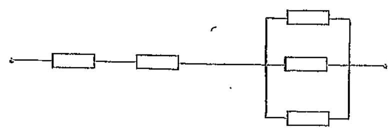
(b) Energy dissipated in time t,

$$0.45 \times 10^3\, t = 12\,(0.50 \times 10^6) \qquad 1$$

Giving $\underline{t = 13.3 \times 10^3\,s}$ 1

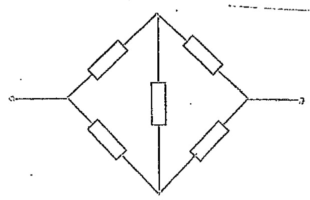
$= 3.7$ hours 2

(c) (i)



1

(ii)



2

Q1

(c) Cont. Total resistances given by

(i) $$R_T = 2R + \left(\frac{1}{R}+\frac{1}{R}+\frac{1}{R}\right)^{-1}$$

$$R_T = \frac{7}{3}R$$ 1

(ii) This is a balanced Wheatstone bridge with no current through central resistor, ie 2R in parallel with 2R, giving

$$R_T = \left(\frac{1}{2R}+\frac{1}{2R}\right)^{-1}$$

$$R_T = R$$ 1

5

(d) At the instant of collision, with speeds v_1 and v_2, each sphere has a gravitational potential

$V = \frac{Gm_1m_2}{(r_1+r_2)}$, momentum conservation $m_1v_1 = m_2v_2$ 1+1 = 2

Using conservation of energy for each sphere with speeds v_1 and v_2, respectively, on impact

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{Gm_1m_2}{(r_1+r_2)}$$ 1

Sub. $v_2 = v_1\left(\frac{m_1}{m_2}\right)$, (1 mark for working)

$$v_1^2 = \frac{2Gm_2^2}{(r_1+r_2)(m_1+m_2)}$$ 1+1 = 2

Giving

$$v_1 = \sqrt{\frac{2Gm_2^2}{(r_1+r_2)(m_1+m_2)}} \quad \& \quad v_2 = \sqrt{\frac{2Gm_1^2}{(r_1+r_2)(m_1+m_2)}}$$ 1

As speeds in opposite directions, relative speed, V_{Rel}, is given by

$$V_{Rel} = \sqrt{\frac{2G(m_1+m_2)}{(r_1+r_2)}}$$ 1

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(e) Assume k mols injected into tyre at each stroke. 1

After n strokes, pressure P_n and volume V_n satisfy

$$P_nV_n = nkRT$$ 1

Final pressure & volume: $P_n = 3.00\times10^5$ Pa & $V_n = 1.2\times10^{-3}$ m^3 1

So $(3.00\times10^5)(1.2\times10^{-3}) = nkT$ (1) 1

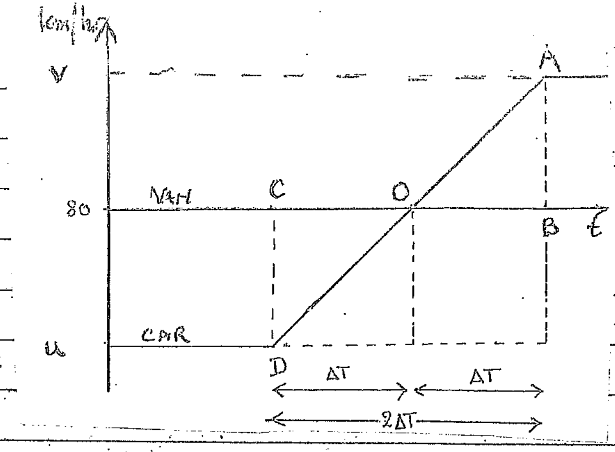
For the first stroke $(1.0\times10^5)(9\times10^{-5}) = kRT$ (2) 1

From (1) and (2) $$n = \frac{(3.00\times10^5)(1.2\times10^{-3})}{(1.0\times10^5)(9\times10^{-5})} = 40$$ 1

5

Q1

(f)



GRAPHICAL SOLUTION

$$80\ km/hr = \frac{80\times10^3}{60\times60}\ ms^{-1} = 22.22\ ms^{-1}$$ 1

Using notation in the diagram, time taken for van to go from C to B

$$2\Delta T = \frac{0.50\times10^3}{22.22}\ s$$

$$\underline{2\Delta T = 22.5\ s}$$ 1

Gradient of OA gives

$$1.2 = \frac{V-22.22}{\Delta T} = \frac{V-22.22}{11.25}$$ 1

Giving $V = 1.2(11.25) + 22.22$

$$\underline{V = 35.72\ ms^{-1}\ \text{or}\ 128.6\ km/hr}$$ 1

4

ALGEBRAIC SOLUTION

$2\Delta T = 22.5\ s$ (see previous derivation) 2

$$V = u + 1.2(2\Delta T)$$
$$= u + 1.2(22.5)$$
$$\underline{V = u + 27.0} \quad (1)$$

Using '$s = ut + \frac{1}{2}ft^2$' with $u = V - 27.0$ 1

$$500 = (V-27.0)(22.5) + \frac{1}{2}(1.2)(22.5)^2$$

Giving $V = 27.0 + 8.72$

$$V = 35.7\ ms^{-1} = 128.6\ km/hr$$ 1

4

Q4

(g) Amount of heat absorbed by calorimeter with water

$Q_1 = 4200(0.80)(10.0) + 42.8(10.0)$

$Q_1 = 34028$ J (34030) 1

Let T_i be the initial temperature of the lead, heat lost to water just before solidification

$Q_2 = (158)(0.40)(T_i - 327)$

$Q_2 = 63.2(T_i - 327)$ cals 1

Heat released by lead during freezing

$Q_3 = (2.323 \times 10^4)(0.40)$

$Q_3 = 9292$ J 1

Heat lost by lead from solidification to 25°C

$Q_4 = (137)(0.40)(327 - 25)$

$Q_4 = 16549.6$ J 1

[Allow for 1 example of e.c.f provided it is not a silly value.]

$Q_1 = Q_2 + Q_3 + Q_4$

Substituting $34028 = 63.2(T_i - 327) + 9292 + 16549.6$

$T_i = 327 + 129.5$

$T_i = 457°C$ 1

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(h) The acceleration is greatest for greatest displacement.
For amplitude A this is $\omega^2 A$
The mass leaves the pan as soon as

$\omega^2 A = g$ 1

For period T, $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.50}$ 1

Substituting, $A = 9.81/(2\pi/0.50)^2$

Giving $A = 6.21$ cm 1

3

Q1.

(i) Charge on the spheres given by "$Q = 4\pi\varepsilon_0 RV$" 1

Thus, using subscripts to indicate the spheres,

$$\left.\begin{aligned} Q_6 &= C_6 V = 6R(4\pi\varepsilon_0)V \\ Q_3 &= 0 \\ Q_2 &= C_2 V = 2R(4\pi\varepsilon_0)V \end{aligned}\right\} \quad 1$$

(Total charge $4\pi\varepsilon_0 V(8R)$)

All spheres have the same final potential, V_f, after touching with wire. Charge conservation requires

$$V_f(4\pi\varepsilon_0)(6R+3R+2R) = Q_6 + Q_2 \quad 1$$
$$= 4\pi\varepsilon_0 V(8R)$$

Giving $\underline{V_f = \frac{8}{11}V}$ 1

Charge on 3R sphere $= 4\pi\varepsilon_0(3R)V_f = 4\pi\varepsilon_0(3R)\frac{8}{11}V$

Fraction of original charge $= \dfrac{4\pi\varepsilon_0 VR\left(\frac{24}{11}\right)}{4\pi\varepsilon_0 VR(8)} = \dfrac{3}{11}$ 1

5

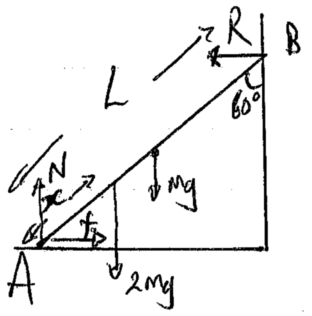
(j) Conservation of energy gives

$$"hf = \tfrac{1}{2}mv^2 + W" \quad 1$$

$$\therefore W = hf - \tfrac{1}{2}mv^2$$
$$= 6.63\times10^{-34}\ \frac{3.00\times10^8}{248\times10^{-9}} - 8.60\times10^{-20}\ \text{J} \quad 1$$
$$= 8.02\times10^{-19} - 8.60\times10^{-20}$$
$$\therefore \underline{W = 7.16\times10^{-19}\ \text{J}} \quad 1$$

3

(k)



Reaction, R, at B. Zero friction.
friction, f, at A
Normal reaction, N, at A.

DIAGRAM OF FORCES ____ 1

Resolving forces vertically: $N = 3mg$
" " horizontally: $f = R$ } 1

and $f = 0\cdot4 \times N = 0\cdot4 \times 3mg$

Take Moments about A: $L \cdot R \cos 60° = mg \frac{L}{2} \cdot \cos 30° + 2mg\,x \cdot \cos 30°$ 1

$(\div L)$ i.e. $R \cdot \frac{1}{2} = mg \frac{\sqrt{3}}{4} + \frac{x}{L} \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}}$

$R = 2mg\sqrt{3}\left(\frac{1}{4} + \frac{x}{L}\right)$ 1

Slipping occurs when $f < R$.
Limit is when $f = R$ i.e. $0\cdot4 \times 3mg = 2mg\sqrt{3}\left(\frac{1}{4} + \frac{x}{L}\right)$

Simplifying $\frac{x}{L} = 0\cdot2\sqrt{3} - \frac{1}{4}$

$= 0\cdot0964$

$\frac{x}{L} = 9\cdot6\%$ 1

5

Alternatively, Take moments about B

$$mg \cdot \frac{L}{2} \cdot \cos 30° + 2mg(L - x)\cos 30° + f \cdot L \cdot \cos 60° = 3mg\,L \cdot \cos 30$$ (1+1)

$(\div mgL)$ $\frac{\sqrt{3}}{4} + \frac{(L-x)}{L}\sqrt{3} + \frac{f}{mg} \cdot \frac{1}{2} = 3\frac{\sqrt{3}}{2}$

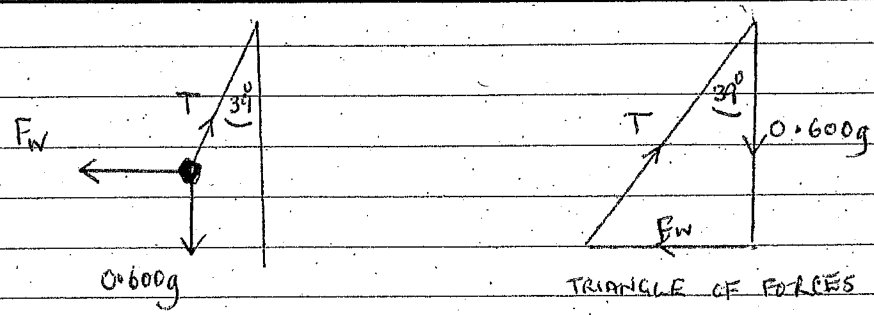
$(\div\sqrt{3})$ $\frac{1}{4} + \left(1 - \frac{x}{L}\right) + \frac{f}{mg} \cdot \frac{1}{2} = \frac{3}{2}$

$\frac{1}{4} + 1 - \frac{x}{L} + 0\cdot4\frac{\sqrt{3}}{2} = \frac{3}{2}$

$\frac{x}{L} = 0\cdot0964$

$= 9\cdot6\%$ (1)

(l)



Triangle of forces gives

$$\frac{F_W}{0{\cdot}600g} = \tan 39^\circ \qquad 1$$

$$\underline{F_W = 4{\cdot}77 \text{ N}} \qquad (1) \qquad 1$$

This is equal to the rate of change of the momentum of the wind.

If wind, speed v, is reduced to rest by ball

$$F_W = \pi\,(0{\cdot}10)^2(1{\cdot}293)\,v^2 \qquad (2) \qquad 1$$

From (1) and (2) $\quad v^2 = \dfrac{4{\cdot}77}{\pi\,(0{\cdot}01)(1{\cdot}293)}$

$= 117{\cdot}3$

$$\underline{v = 10{\cdot}8 \text{ ms}^{-1}} \qquad 1$$

4

(m)

$$\frac{1}{8} = \exp(-420\ln 2/T) \qquad (T \text{ in days}) \qquad 1$$

T is half life in days

Thus $\quad 2{\cdot}208 = 0{\cdot}6931\,(420)\,/T$

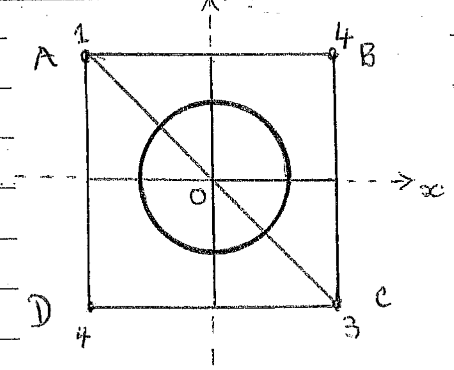
Giving $\quad \underline{T = 140 \text{ days}} \qquad 1$

a = 210
b = 84
c = 4
d = 2
e = 0
f = 0

Subtract one mark, from 2 marks, for each error – two or more errors gives zero marks $\qquad 2$

4

(n)



Q1

(m) Cont. The two 4kg masses and the mass 8kg of the ring are equivalent to a mass of 16 kg at origin O.
Taking moments about O for the 1, 16 and 3 kg masses, the centre of mass lies along AC below O at a distance $\bar{D}$, from O, given by 1

$$(1+16+3)\bar{D} = 3\sqrt{2}b - 1\sqrt{2}b$$ 1

$$\therefore \quad 20\bar{D} = 2\sqrt{2}b$$

$$\bar{D} = \frac{\sqrt{2}}{10}b$$ 1

3

Alternative solutions acceptable.

(o) V_{car} vel. of car, freqs. f. $f_{466} = \frac{c}{(c - V_{car})} f_{440}$ $(c = 331\ ms^{-1})$ 1

Substituting $f_{466} = 466$ and $f_{440} = 440$

$$466 = \frac{331 \times 440}{331 - V_{car}}$$ 1

$$V_{car} = 18.5\ ms^{-1}$$ 1

3

(p) Let v be velocity of protons, mass m_p, then energy conservation gives

$$\frac{1}{2}m_p v^2 = 2000\,e$$

$$v = \sqrt{\frac{4\times10^3 e}{m_p}} \quad (1)$$ 1

Radius of path in magnetic field B given by R_p where

$$\frac{m_p v^2}{R_p} = BeV$$ 1

$$R_p = \frac{m_p V}{Be} = \frac{m_p}{Be}\sqrt{\frac{4\times10^3 e}{m_p}} \quad (2)$$

$$= \frac{1}{0.200}\sqrt{\frac{(4\times10^3)(1.67)10^{-27}}{1.60\times10^{-19}}}$$

$$R_p = 3.23\text{cms}$$ 1

Deuterons have mass $2M_p$. Thus from (2) their radius Rd is given by $Rd = R_p\sqrt{2}$

$$Rd = 4.57 = 4.6\ cm.$$ 1

4

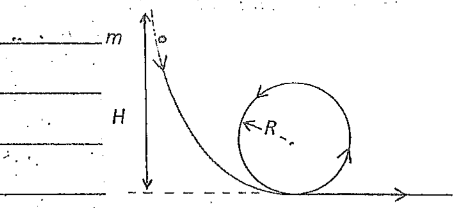
~~Deuterons have mass 2m_p. Thus from (2) their radius R_d given by~~

$$\cancel{R_d = R_p\sqrt{2}}$$

~~Substituting~~

$$\cancel{R_d = 4.5\ \text{cm}}$$

(9)



If the mass is to complete the circular path, then at the top of the circle we require the reaction of the constraining circle, R_c, to satisfy $R \geq 0$

Equation for circular motion

$$\frac{mv^2}{R} = R_c + mg$$

For $R_c \geq 0$

$$\frac{mv^2}{R} \geq mg \qquad 1$$

The smallest v occurs when

$$v^2 = Rg \qquad (1)$$

However from the conservation of energy

$$\frac{1}{2}mv^2 = mg(H - 2R) \qquad 1$$

Giving from (1)

$$Rg = g(H - 2R)$$

Smallest value of H given by

$$\underline{H = \tfrac{5}{2}R} \qquad 1$$

3

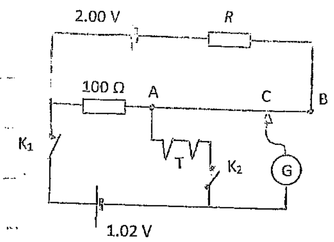
Q2

(a) (i) $\varepsilon = \varepsilon_1 \cdot \frac{l_1}{l}$ 1

$\frac{\varepsilon_2}{l_2} = \frac{\varepsilon_1}{l_1}$ 1

$\frac{\varepsilon_3}{l - l_2} = \frac{\varepsilon_1}{l} \Rightarrow \frac{\varepsilon_3}{\varepsilon_1} = \left(1 - \frac{l_2}{l}\right)$

(b)



No current flows in lower arm. 1

Let i be current in upper circuit then

$(R+102)\,i = 2.00$ as AB has resistance 2Ω (1) 1

With K_1 closed and K_2 open, equating pds

$1.02 = i\,(100 + 2 \times 0.90)$ 1

Giving $i = \frac{1.02}{101.8}$ (2) 1

K_1 open and K_2 closed, equating pds

$\varepsilon = i\,(2 \times 0.45) = i\,(0.90)$ 1

Thus from (2) $= \frac{1.02}{101.8}\,(0.90)$

$\underline{\varepsilon = 9.02 \times 10^{-3}\ V}$ 1

From (1)

$R = \frac{2.00}{i} - 102$ 1

$= 2.00\ \frac{101.8}{1.02} - 102$ from (2)

$\underline{R = 97.65\ \Omega}$ 1

8

Q2 (c) Since the bridge is balanced, $\frac{V}{I}$ (= bulb resistance) = $\underline{4\,\Omega}$ 1

Given $V = 2I + 8I^2 \Rightarrow \frac{V}{I} = 2 + 8I$ 1

$\therefore\ 4 = 2 + 8I \Rightarrow I = \frac{1}{4}\,A$ 1

Hence $V = 4\,\Omega \times \frac{1}{4}\,A = 1\,V$

$\therefore\ \underline{V_b = 2V}$ $\frac{1}{4}$

Alternatively: For a balanced bridge, all resistors are $4\,\Omega$. 1

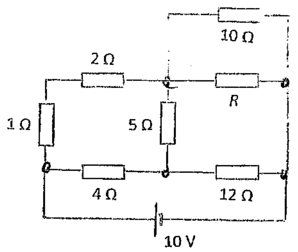
So $V = \frac{V_b}{2}$ 1

and with $8\,\Omega$ per arm, $I = \frac{V_b}{8}$ 1

Substituting $\frac{V_b}{2} = 2\left(\frac{V_b}{8}\right) + 8\left(\frac{V_b}{8}\right)^2 \Rightarrow V_b = 2$ Volts 1

(change from paper marks) $\overline{4}$

Q2 (d)



This circuit is a Wheatstone bridge. Alternatively symmetry requires "$\frac{R_1}{R_2} = \frac{R_3}{R_4}$" 1

Simplification gives

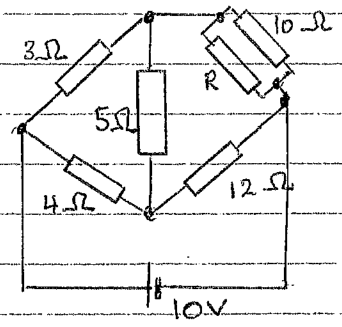


Diagram 1

For minimum heat generated in $5\,\Omega$, current in $5\,\Omega$ is zero. 1

Thus the bridge must be 'balanced', by symmetry with "$\frac{R_1}{R_2} = \frac{R_3}{R_4}$".

This requires

$$\frac{1}{12}\left(\frac{1}{R} + \frac{1}{10}\right)^{-1} = \frac{3}{4}$$ 1 + 1

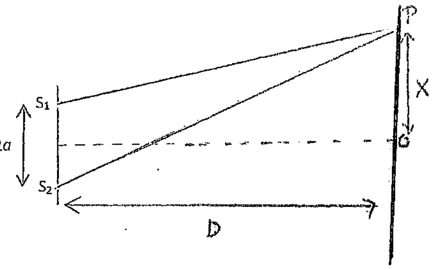
$$\frac{1}{R} + \frac{1}{10} = \frac{1}{9}$$

$$\frac{1}{R} = \frac{1}{90}$$

$$\underline{R = 90\,\Omega}$$ $\frac{1}{6}$

(change from paper marks)

Q3 (a)



METHOD I

	$S_2P = \left[D^2 + (x+a)^2\right]^{1/2}$	1
	$= D\left[1 + \left(\frac{x+a}{D}\right)^2\right]^{\frac{1}{2}}$	1
Expanding	$= D\left[1 + \frac{1}{2}\left(\frac{x+a}{D}\right)^2 + \cdots\right]$	1
Similarly	$S_1P = D\left[1 + \frac{1}{2}\left(\frac{x-a}{D}\right)^2 + \cdots\right.$	1
Thus		
	$S_2P - S_1P = \frac{1}{2D}\left((x+a)^2 - (x-a)^2\right) + \cdots$	1
Giving	$\underline{S_2P - S_1P = \frac{2ax}{D}} + \cdots$	1
		6

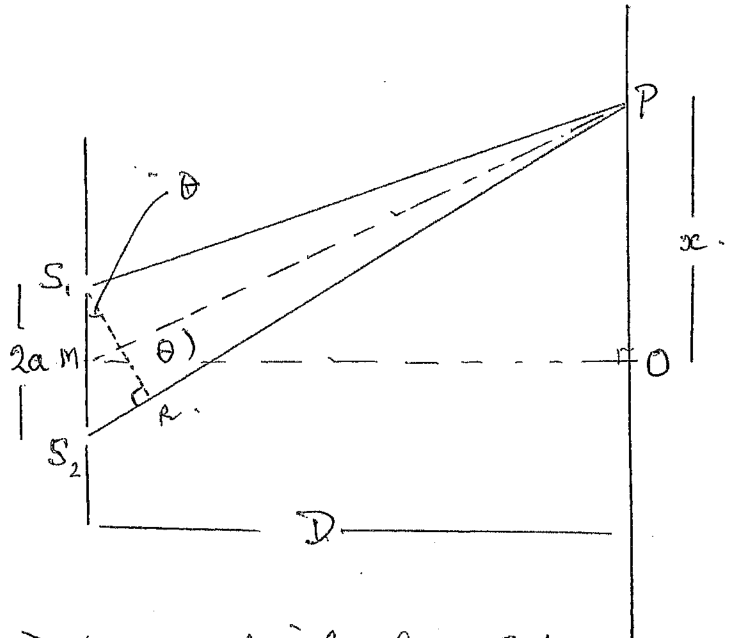
METHOD II

	$S_2P^2 - S_1P^2 = \left[D^2 + (x+a)^2\right] - \left[D^2 + (x-a)^2\right]$	2
	$= 4ax$	2
	$\therefore (S_2P - S_1P)(S_2P + S_1P) = 4ax$	1
	$\underline{S_2P - S_1P = \frac{2ax}{D}}$ as $S_2P + S_1P \simeq 2D$	1
		6

For fringes	$S_2P - S_1P = n\lambda$ where n is an integer	1
Spacing between adjacent fringes is λ	ie optical path difference	1
or from above	$\frac{2a}{D}\Delta x = \lambda$	1
Giving	$\underline{\Delta x = \frac{\lambda D}{2a}}$ (1)	1
		4

METHOD III

Alternative Derivation



Drop perpendicular from S_1 to S_2P, intersecting S_2P at R.

$O\hat{M}P = \theta$. 1

Now S_1S_2 ~~perpendicular~~ to MO and, to a good approx, as θ small, MP perpendicular S_1R.

Thus $R\hat{S_1}S_2 = \theta$ 1

Consequently Δs S_2S_1R and POM similar, so that

$S_2\hat{S_1}R = \theta$. 1

Optical path difference

$$\phi = 2a \sin\theta \qquad 1$$
$$\simeq 2a \tan\theta \qquad \text{as } \theta \text{ small} \qquad 1$$
$$= \frac{2ax}{D}$$
$$\phi = \frac{2ax}{D} \qquad 1$$

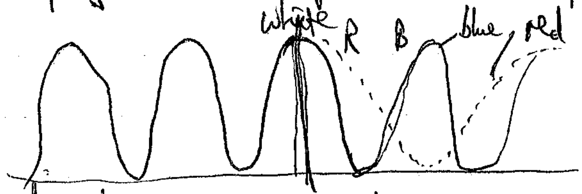
6

Then equate to $n\lambda$ as previous page.

Q.3

(a) (ii) White light fringes occur when the optical path difference for all wavelengths is zero or close to zero; giving a white or whitish fringe at the centre. 1

Coloured fringes either side of the white central fringe 1



sketch (1) 2

(iii) The bulb is an incoherent source; individual photons have no fixed/constant phase difference. 1

The single slit diffracts each photon to cover the pair of slits so that the slits act as coherent sources for every photon 1

2

14

Q3 (b) Using result (1) obtained for light waves and applying it to sound waves 1

$$\text{"}\ \Delta x = \frac{\lambda D}{2a}\ \text{"}$$ 1

Substituting numerical values

$$1{\cdot}14 = \lambda \frac{20{\cdot}0}{3{\cdot}0}$$ 1

$$\lambda = \frac{(1{\cdot}14)(3{\cdot}00)}{20{\cdot}0}$$ 1

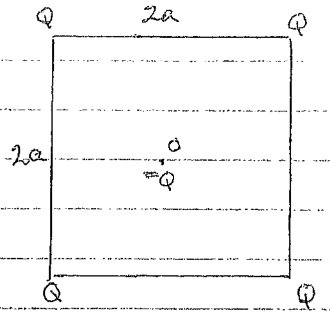
Velocity of sound "$c = f\lambda$" 1

$$c = (2\times10^{3})\frac{(1{\cdot}14)(3{\cdot}00)}{20{\cdot}0}\ ms^{-1}$$

Giving $\underline{c = 342\ ms^{-1}}$ 1

6

Q4(a) (i)



The total PE is the sum of the PEs for each pair of charges.

Total PE for 4 edges $= 4\dfrac{Q^2}{4\pi\varepsilon_0(2a)}$ (1) 1

Total PE for $-Q$ and Qs at the four corners

$= -4\dfrac{Q^2}{4\pi\varepsilon_0(a\sqrt{2})}$ (2) 1

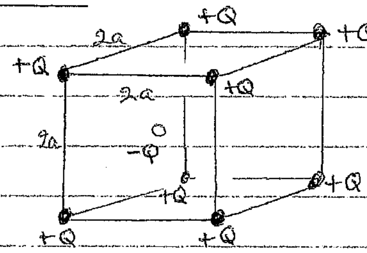
Total PE for diagonal Qs

$= 2\dfrac{Q^2}{4\pi\varepsilon_0\sqrt{2}(2a)}$ (3) 1

Thus the total PE, V_s, given by

$$V_s = \frac{Q^2}{4\pi\varepsilon_0 a}\left[2+\frac{1}{\sqrt{2}}-\frac{4}{\sqrt{2}}\right] \rightarrow = \frac{-Q^2}{4\pi\varepsilon_0 a}(0{\cdot}121)$$

$$V_s = \frac{Q^2}{4\pi\varepsilon_0 a}\left(2-\frac{3\sqrt{2}}{2}\right) = \frac{-Q^2}{4\pi\varepsilon_0 a}\left(\frac{3\sqrt{2}}{2}-2\right)$$ 1



(ii) PE due to $(-Q)$ $= -8\dfrac{Q^2}{4\pi\varepsilon_0 a\sqrt{3}}$ 1

P.E due to body diagonals $= 4\dfrac{Q^2}{4\pi\varepsilon_0 2a\sqrt{3}}$ 1

P.E due to face diagonals $= 12\dfrac{Q^2}{4\pi\varepsilon_0 2a\sqrt{2}}$ 1

P.E. due to edges $= 12\dfrac{Q^2}{4\pi\varepsilon_0 2a}$ 1

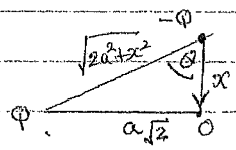
(do not count edges twice as the occur on two faces)

~~Total PE~~, $V_c = \dfrac{Q^2}{4\pi\varepsilon_0 a}\left[6+\dfrac{6}{\sqrt{2}}+\dfrac{2}{\sqrt{3}}-\dfrac{8}{\sqrt{3}}\right]$

$$V_c = \frac{Q^2}{4\pi\varepsilon_0 a}\left[6+3\sqrt{2}-2\sqrt{3}\right] = \frac{Q^2}{4\pi\varepsilon_0 a}(6{\cdot}78)$$ 1

9

b (i) Force towards centre, O, of square



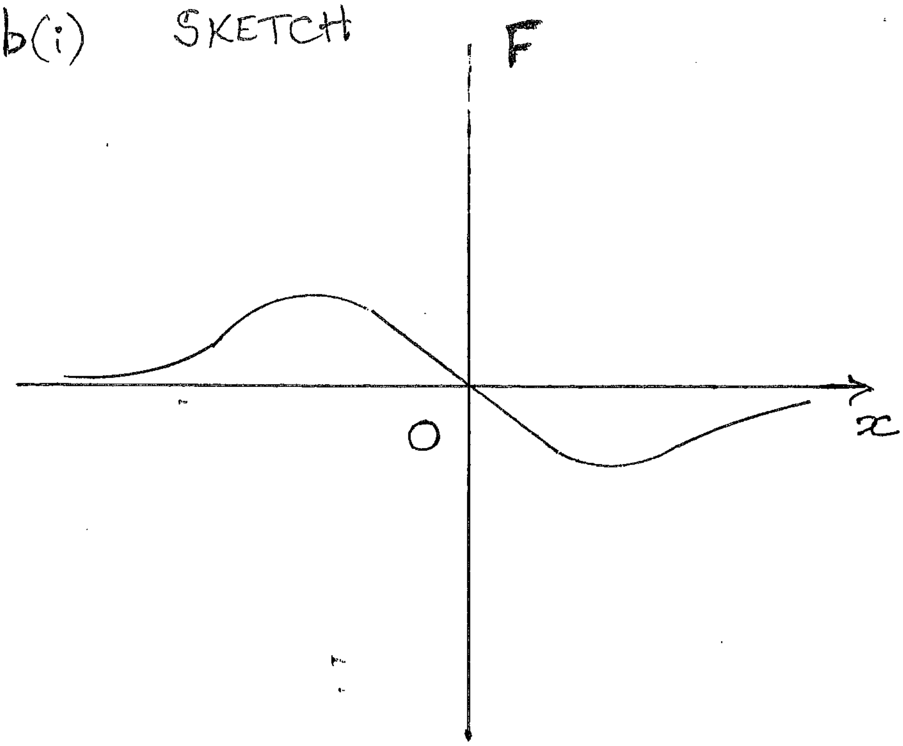
$$\mathbf{F} = 4\frac{(Q^2)}{4\pi\varepsilon_0(2a^2+x^2)}\cos\theta$$ 1

$$= \frac{4Q^2}{4\pi\varepsilon_0(2a^2+x^2)}\frac{x}{\sqrt{(2a^2+x^2)}}$$ 1

$$\mathbf{F} = \frac{4Q^2}{4\pi\varepsilon_0}\frac{x}{(2a^2+x^2)^{3/2}}$$ (negative sign for force in opposite direction) 1

3

Q4 b(i) SKETCH



$F = 0$ at 0 & $\pm\infty$ 1

F has min for $x > 0$ & max. for $x < 0$ 1

3 + 2 = 5

Q4(b)(ii)

Equation of motion

$$m\ddot{x} = -\frac{4Q^2}{4\pi\varepsilon_0}\frac{x}{(2a^2)^{3/2}\left(1+\frac{x^2}{2a^2}\right)^{3/2}}$$ 1

Expanding

$$= -\frac{Q^2}{\pi\varepsilon_0}\frac{x}{2\sqrt{2}a^3}\left(1 - \frac{3}{2}\frac{x^2}{2a^2} + \ldots\right) \quad (2)$$ 1

Thus for small x $$m\ddot{x} = -\frac{Q^2 x}{2\sqrt{2}a^3\pi\varepsilon_0}$$ 1

This is the equation of SHM with period

$$T = 2\pi\sqrt{\frac{2\sqrt{2}a^3\pi\varepsilon_0 m}{Q^2}} = 2\pi\sqrt{\frac{(4\pi\varepsilon_0)\sqrt{2}a^3\pi m}{2Q^2}}$$ 1

From (2) we require for SHM that when $x = A$

$$\frac{3}{2}\frac{A^2}{2a^2} \ll 1$$ 1

$$A^2 \ll 4a^2/3$$

ie. $$A \ll \frac{2\sqrt{3}a}{3}$$ 1

6

Q.5 Galaxy Rotation Curve

(a) Equating the gravitational acceleration with the centripetal acceleration:

$$\frac{G M(<r)}{r^2} = \frac{V_{rot}^2(r)}{r} \quad [1]$$

$$\therefore \quad V_{rot}^2(r) = \frac{G M(<r)}{r}$$

$$\therefore \quad V_{rot}(r) = \sqrt{\frac{G M(<r)}{r}} \quad [1]$$

Total: 2 marks

(b) i) $r < r_0$ (inside the bulge)

$$\rho_0 = \frac{M(<r)}{\frac{4}{3}\pi r^3} \text{ - constant} \Rightarrow M(<r) = \frac{4}{3}\pi r^3 \rho_0 \quad [1]$$

$$\therefore \; V_{rot}(r) = \sqrt{\frac{4\pi \rho_0 G}{3}} \cdot r \Rightarrow V_{rot}(r) \propto r \text{ at } r < r_0 \; [1]$$

ii) $r > r_0$ (outside the bulge)

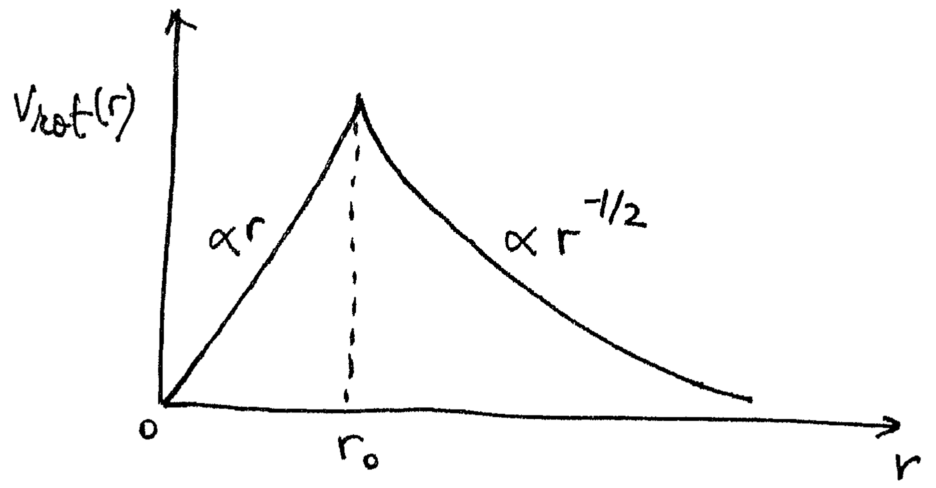
If most of the mass is enclosed in the bulge:

$$M(<r) = M = \text{constant} \quad [1]$$

$$\therefore \; V_{rot}(r) = \sqrt{GM} \cdot r^{-1/2} \Rightarrow V_{rot}(r) \propto r^{-1/2} \text{ at } r > r_0 \; [1]$$

$$\Rightarrow \quad V_{rot}(r) \propto \begin{cases} r, & r < r_0 \\ r^{-1/2}, & r > r_0 \end{cases}$$

Q5 The rotation curve of the galaxy:



1 mark for $r<r_0$

1 mark for $r>r_0$

1 mark for correct labels (v_{rot}, r, r_0)

Total: 7 marks

(c) Dark matter profile: $\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha}$

Velocity curve: $v_{rot}(r) = \sqrt{\frac{GM(<r)}{r}}$

Spherical distribution: $\rho(r) = \frac{M(r)}{\frac{4}{3}\pi r^3} = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha}$ [1]

(See the integration for M(r) on the next page – the exponent will be the same for dimensional reasons. So this simplistic method is OK) [1]

$$\Rightarrow M(<r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha} \cdot \frac{4}{3}\pi r^3$$

$$\therefore v_{rot}(r) = \sqrt{G\rho_0 \left(\frac{r}{r_0}\right)^{-\alpha} \cdot \frac{4}{3}\pi r^2}$$

$$\therefore v_{rot}(r) = \sqrt{\frac{4\pi G \rho_0 r_0^{\alpha}}{3}} \cdot r^{1-\alpha/2}$$ [1]

Flat velocity profile at $r>r_0 \Rightarrow v_{rot}(r>r_0) = \text{constant}$

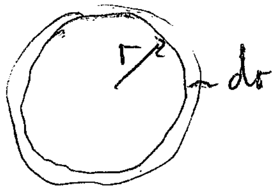
$$\Rightarrow 1 - \alpha/2 = 0 \Rightarrow \underline{\alpha = 2}$$ [1]

$$\Rightarrow \rho(r) \propto r^{-2}$$

Total: 4marks

part (c) alternative

Knowing the density profile, we can calculate the mass within radius r (still using the inverse square law of force so that the Gaussian assumption is still valid).



$$M(r) = \int 4\pi r^2 \, dr \, \rho(r) \quad [1]$$

$$= \int 4\pi r^2 \rho_0 \frac{r^{-\alpha}}{r_0^{-\alpha}} \, dr$$

$$= \frac{4\pi\rho_0}{r_0^{-\alpha}} \frac{r^{3-\alpha}}{(3-\alpha)} . \quad [1]$$

Velocity curve $V_{rot}(r) = \sqrt{\frac{G M(r)}{r}}$

$$= \sqrt{G \frac{4\pi\rho_0}{r_0^{-\alpha}} \frac{r^{2-\alpha}}{3-\alpha}}$$

$$= \sqrt{\frac{G\, 4\pi\rho_0}{r_0^{-\alpha}(3-\alpha)}} \; r^{1-\frac{\alpha}{2}} \quad [1].$$

$$1 - \frac{\alpha}{2} = 0 \Rightarrow \underline{\underline{\alpha = 2}} \quad [1]$$

4 marks

[With this simple model we assume that the dark matter dominates outside r_0. Otherwise we have a term like
$k_1 r^{-1} + k_2 r^{2-\alpha} = k_3 r^0$ for $r > r_0$
so require $r \gg r_0$
to give $\alpha = 2$.]

Q5

(d) $$V_{rot}(r) = \sqrt{\frac{GM(<r)}{r}}$$

$$\therefore M(<r) = \frac{V_{rot}(r)^2 \cdot r}{G}$$

$r = 2.8 \times 10^5$ light years $= 2.65 \times 10^{18}$ km

(i) $v_{rot}(r) = 220 \text{ kms}^{-1}$

$$\therefore M_{tot} = 1.9 \times 10^{42} \text{kg} = \underline{9.7 \times 10^{11} M_\odot}$$ [1]

ii) $V_{rot}(r) = V_{exp}(r) = 70 \text{ kms}^{-1}$

$$\therefore M_{visible} = 1.95 \times 10^{41} \text{kg} = \underline{9.8 \times 10^{10} M_\odot}$$ [1]

Dark matter fraction: $f_{DM} = \frac{M_{tot} - M_{visible}}{M_{tot}}$

$\therefore \underline{f_{DM} \simeq 0.9}$ (90% Dark matter) [1]

Total: 3 marks

(e) In the small acceleration (MOND) approximation:

$$\mu\left(\frac{a}{a_0}\right) = \frac{a}{a_0}$$

$$\therefore F = m\mu\left(\frac{a}{a_0}\right)a = \frac{ma^2}{a_0}$$ [1]

(The force is proportional to the acceleration squared)

a is the centripetal acceleration

$$\therefore a = \frac{V_{rot}^2}{r}$$ [1]

For an object of mass m in circular orbit around a point mass M (same as the case for $r > r_0$ in b)):

Q5

$$\frac{GM\cancel{m}}{\cancel{r^2}} = m\frac{a^2}{a_0} = \cancel{m}\frac{\left(\frac{V_{rot}^2}{\cancel{r}}\right)^2}{a_0} \quad [1]$$

$$\therefore \quad V_{rot}^4(r) = GMa_0$$

$$\therefore \quad \underline{V_{rot}(r)} = \sqrt[4]{\underline{GMa_0}} = \underline{\text{constant}} \quad [1]$$

$\therefore$ the rotation curve is flat at $\underline{r > r_0}$.

Total: 4 marks

Q6 (a) (i) Equating R^α/T^γ for the Earth and Jupiter

$$\frac{(1.50\times10^{11})^\alpha}{1^\gamma} = \frac{(7.76\times10^{11})^\alpha}{(11.8)^\gamma} \quad \text{T in years} \quad 1$$

So

$$\left(\frac{7.76}{1.50}\right)^\alpha = (11.8)^\gamma$$

Taking logs,

$$\alpha \log\left(\frac{7.76}{1.50}\right) = \gamma \log(11.8) \quad 1$$

$$\alpha(0.7138) = \gamma(1.0719)$$

Giving

$$\frac{\alpha}{\gamma} = 1.50 \quad 1$$

If M_S is the mass of the Sun, M_E is the mass of the Earth and R_{SE} the radius of the orbit of the Earth, then for circular motion

$$\frac{GM_EM_S}{R_{SE}^2} = \frac{M_EV_E^2}{R_{SE}} \quad V_E \text{ is speed of Earth} \quad 2$$

$$M_S = \frac{R_{ES}V_E^2}{G} \quad 1$$

As $V_E = 2\pi(1.50\times10^{11}) / 365\times24\times60\times60$ 1

$$M_S = \frac{(1.50\times10^{11})}{(6.67\times10^{-11})}\left[\frac{2\pi(1.50\times10^{11})}{365\times24\times60\times60}\right]^2 \quad 1$$

$$M_S = 2.01\times10^{30}\text{ kg} \quad \text{-(this is given)} \quad 0$$

8

(b)

$$V_S = \frac{2\pi R_s}{24\times60\times60} \quad (1)$$

Equation of motion for satellite mass m_S.

$$\frac{m_SV_S^2}{R_S} = \frac{Gm_SM_E}{R_S^2} \quad 1$$

Substituting for V_S from (1) gives

$$R_S^3 = GM_E\left(\frac{24\times60\times60}{2\pi}\right)^2 \quad 1$$

$$= (6.67\times10^{-11})(5.98\times10^{24})\left(\frac{24\times60\times60}{2\pi}\right)^2$$

$$= 7.54\times10^{22}$$

$$R_S = 4.22\times10^7\text{ m} = 4.22\times10^4\text{ km} \quad 1$$

Giving from (1)

$$V_S = 3.07\times10^3\text{ ms}^{-1} = 3.07\text{ kms}^{-1} \quad 1$$

4

Q6 (c)

(c)(i) For a space craft mass m_s the escape velocity is given by, equating ke and pe,

(ii)
$$\frac{1}{2} m_s V_E^2 = \frac{G M_E m_s}{R_E}$$ 1

M_E is the and R_E the radius of the Earth

Giving
$$V_E = \sqrt{\frac{2GM_E}{R_E}}$$ 1
$$= \sqrt{2(6.67\times10^{-11})(5.98\times10^{24})/(6.38\times10^{6})}$$
$$V_E \approx 11.2 \quad kms^{-1}$$ 1

(iii) If one launches the spacecraft in the direction of the Earth's rotation at the equator, one starts, before the rocket is ignited, with an initial ke. 1

(iv) The velocity of the Earth's rotation at the equator, V_R, is given by
$$V_R = \frac{2\pi\,(6.38\times10^{6})}{24\times60\times60}$$
$$= 4.64\times10^{2} \quad ms^{-1}$$ 1

Thus the minimum initial launch speed V_i is given by
$$V_i = 11.2 - 0.46 \quad kms^{-1}$$
$$V_i \approx 10.7 \quad kms^{-1}$$ 1

6

(d)
$$V = \sqrt{\frac{2GM_s}{r_{orbit}}} = 42.2 \text{ km/s}$$ 1

Orbital speed of Earth
$$= \frac{2\pi \; 1.5\times10^{11}}{365\times24\times3600}$$ 1
$$= 29.9 \text{ km/s}$$

2

$$\left(V_{min} = 42.2 - 29.9 = 12.3 \text{ km/s}\right)$$

Q7

(a) If m is the mass of the drop and E the field, then for constant velocity, assuming n charges on the drop.

By Newton II, $E n_1 e = mg + 6\pi\eta a u_1$

and $E n_2 e = mg + 6\pi\eta a u_2$ } 3

Subtracting $Ee(n_2 - n_1) = 6\pi\eta a(u_2 - u_1)$ 1

$$\therefore (n_2 - n_1) = \frac{6\pi\eta a}{Ee}(u_2 - u_1)$$

and $E = \dfrac{5000}{1.5\times10^{-2}}$ V/m 1

$$\therefore (n_2 - n_1) = \frac{6\pi(1.82\times10^{-5})(2.76\times10^{-6})}{1.60\times10^{-19}(5000/1.5\times10^{-2})}\left[\frac{10^{-2}}{42} - \frac{10^{-2}}{78}\right]$$

$$= \frac{94.6\times10^{-11}}{5.33\times10^{-14}}\left[2.38\times10^{-4} - 1.28\times10^{-4}\right]$$

$$= (17.7\times10^{3})\left[1.10\times10^{-4}\right]$$ } 2

$(n_2 - n_1) = 1.95$

This is approximately 2 electrons of charge. } 1

8

Q7

(b) Drops coalesce:

$$QE = 6\pi r_1 \eta u_1 + m_1 g \quad (1)$$
$$QE = 6\pi r_2 \eta u_2 + m_2 g \quad (2) \quad \} \; 1$$

Drops Together $\quad 2QE = 6\pi r_3 \eta u_3 + m_3 g \quad (3) \quad 1$

Adding (1) and (2) $\quad 2QE = 6\pi\eta(r_1 u_1 + r_2 u_2) + (m_1+m_2)g$

with $m_3 = m_1 + m_2$

Hence $\quad r_3 u_3 = r_1 u_1 + r_2 u_2 \quad 1$

But volume conserved, so $\frac{4}{3}\pi r_3^3 = \frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 \quad 1$

So $\quad r_3 = \sqrt[3]{r_1^3 + r_2^3}$

Hence $\quad u_3 = \dfrac{r_1 u_1 + r_2 u_2}{\sqrt[3]{r_1^3 + r_2^3}} \quad 1$

$\frac{}{5}$

{ For the general case with different charges Q_1 and Q_2 initially,

$$EQ_1 = 6\pi r_1 \eta u_1 + m_1 g \quad (4)$$
$$EQ_2 = 6\pi r_2 \eta u_2 + m_2 g \quad (5)$$

Drops together $\quad E(Q_1+Q_2) = 6\pi r_3 \eta u_3 + m_3 g \quad (6)$

with $m_3 = m_1 + m_2$

Adding (4) and (5)

$$E(Q_1+Q_2) = 6\pi\eta(r_1 u_1 + r_2 u_2) + (m_1+m_2)g$$

Hence $\quad r_3 u_3 = r_1 u_1 + r_2 u_2$ as above }

Q7 (c) Initial velocity v given by

$$eV = KE \qquad 1$$

$$10^4 e = \tfrac{1}{2} m_e v^2 \qquad 1$$

$$\frac{e}{m_e} = \frac{10^4 v^2}{2} \qquad 1$$

If the magnetic force balances the force due to the electric field

$$evB = eE \qquad 1$$

So $vB = E$

$$(2.5 \times 10^{-3}) v = \frac{3000}{2 \times 10^{-2}}$$

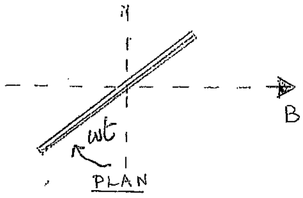
$$v = \frac{3000}{(2\times 10^{-2})(2.5\times 10^{-3})} \qquad 2$$

$$\therefore \frac{e}{m_e} = \frac{10^4}{2} \times 36 \times 10^{14}$$

$$= 1.8 \times 10^{11}\ \mathrm{C/kg} \qquad 1$$

7

Q(8) (a)



$$\Phi = \pi r^2 B \cos\omega t$$ (Alternative notation with $\omega' t = 90 - \omega t$ OK providing subsequent notation consistent) 2

(b) $I = \dfrac{V}{R}$ 1

$I = \dfrac{\pi r^2 B\omega}{R} \sin\omega t$ as $V = -\dfrac{d\Phi}{dt} = \pi r^2 B\omega \sin\omega t$ (1) 2

3

(c) The current in the ring induces a magnetic field at the centre of the ring of magnitude 1

$$B_I = \mu_0 \frac{I}{2r}$$ 1

$$B_I = \mu_0 \frac{\pi r B\omega}{2R} \sin\omega t$$ from (1) 1

3

(d) The direction of B_I is perpendicular to the plane of the ring and rotates with it. Resolving B_I parallel, $B_{\parallel}$, and perpendicular, $B_{\perp}$, to the ring gives

$$B_{\parallel} = B_I \cos\omega t$$ 1

$$= \mu_0 \frac{\pi r B\omega}{2R} \sin\omega t \cos\omega t = \mu_0 \frac{\pi r B\omega}{4R} \sin 2\omega t$$ 1

The average value of $\sin 2\omega t$, over time, is zero 1

$$B_{\perp} = \mu_0 \frac{\pi r B\omega}{2R} \sin^2\omega t$$ 1

$$= \mu_0 \frac{\pi r B\omega}{4R} (1 - \cos 2\omega t)$$ 2

The $\cos 2\omega t$ term averages out to zero leaving

$$B_{\perp} = \frac{\mu_0 \pi r B\omega}{4R}$$ 1

7

(e) Consequently the angle of the compass needle, α, is given by

$$\tan\alpha = \frac{B_{\perp}}{B} = \mu_0 \frac{\pi r\omega}{4R}$$ 1

Giving $R = \mu_0 \pi r \omega / 4\tan\alpha$ 1

Subsituting $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}}$, $\alpha = 2.00°$,
$r = 0.125\ \mathrm{m}$, $\omega = 2\pi(10)\ ^c\mathrm{s^{-1}}$ 2

$$R = 2.22 \times 10^{-4}\ \Omega$$ 1

5